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Concept Of The Month

Inverse Integral

Quick Recap

Excel

Manipulations of Trigonometric Expressions

CHAPTER-WISE PREVIOUS YEAR KVPY Qs

MOCK TEST PAPER (JEE MAIN) - 5,6

FEBRUARY 2019

MATHEMATICS TIMES

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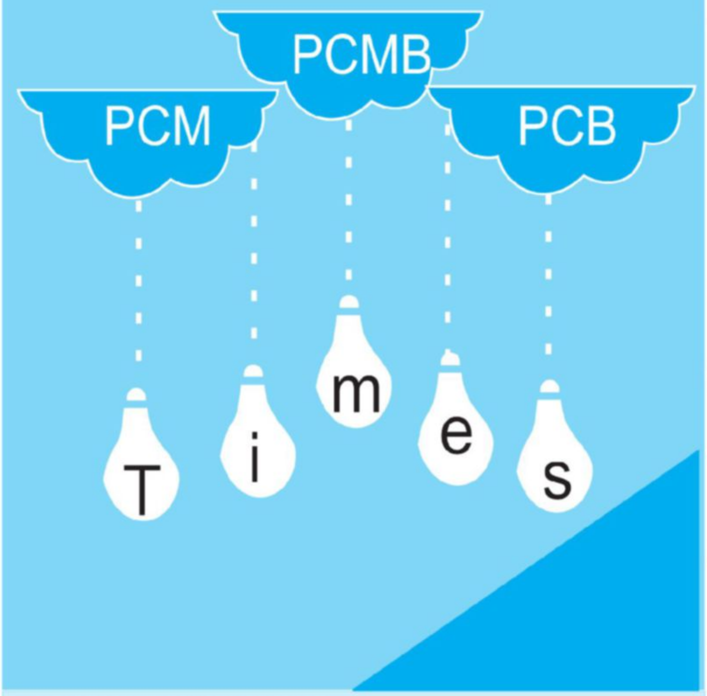
VOL-V



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Inverse Integral

Concept of the month

This column is aimed at preparing students for all competitive exams like JEE, BITSAT etc. Every concept has been designed by highly qualified faculty to cater to the needs of the students by discussing the most complicated and confusing concepts in Mathematics.

By. **DHANANJAYA REDDY THANAKANTI**
(Bangalore)

Introduction

This article explains us to find a indefinite and definite integral of an inverse function when we are known the parent function , like finding integral of $\ln x$ without knowing how to integrate $\ln x$ directly. It can be used to review knowledge about the inverse functions e^x and $\ln x$ and to discuss how to find the area between a curve and the y-axis. This method can be extended to other functions such as $\arcsin x$, Once student can integrate $\sin x$.

If f and f^{-1} are elementary on some closed interval, then integral $f(x)dx$ is elementary i integral $f^{-1}(x)dx$ is elementary.

Take a look at these graphs

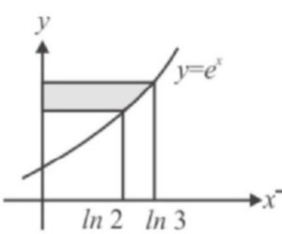


Fig .1

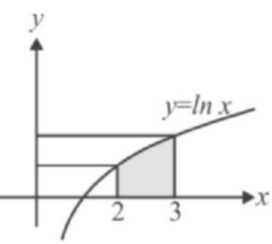
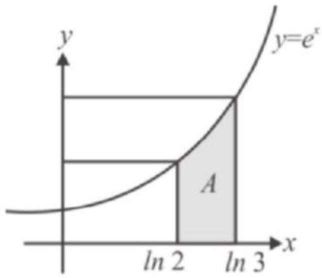


Fig. 2

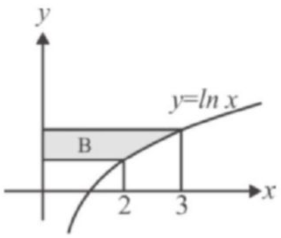
Here are some things we have noticed .

- There are two dierent graphs : each has a function and two x-coordinates given . There is an area shaded on each graph but they are in different places. One is between the curve and the y-axis and the other between the curve and the x-axis .
- Neither curve has any y- coordinates labelled so we can add these to the diagram . For e^x, the y-coordinates are 2 and 3 . For $y=\ln x$ they are $\ln 2$ and $\ln 3$. So the same numbers appears in both graphs except the x and y-coordinates have swapped over.
- The swapping of coordinates might remind us that there is a relationship between e^x and $\ln x$. They are inverse functions of each other, which we can think about graphically as a reflection in $y = x$.
  This symmetry around $y = x$ is not immediately obvious from the diagrams, as the scaling in the x and y directions is different on each sketch. It is important not make assumptions about shape or symmetry based on sketch graphs.
- There are two rectangles on each graph with areas of $2 \times \ln 2$ and $3 \times \ln 3$ respectively. These areas are the same on both graphs as the x and y coordinates are reflected in $y = x$.
  The symmetry of the graphs implies that the two shaded areas are actually identical as they are reflected across $y = x$. They could be represented by the integral $\int_2^3 \ln x\, dx$ or the integral $\int_2^3 \ln y\, dy$.
- We can also find the area of other regions bounded by the curves . For example , the area A can be represented by the integral

$$\int_2^3 e^x dx,$$

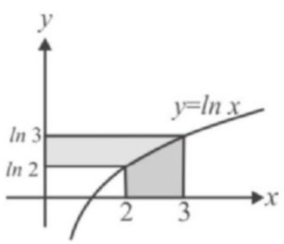


As inverse functions have symmetry around $y = x,$ we know that area A is the same as area B, shaded in the diagram below.



The integral represents the area of the shaded region under the curve $y = ln\, x.$ However, we also know that because these graphs are inverse functions, the shaded region rectangle between e^x and the y- axis has the same area.

If we don't know how to integrate $ln\, x$ directly, then we need to use other areas that we do know how to find. we have already calculated the areas of the large rectangle, $3\,ln\,3$ and the smaller rectangle, $2\,ln\,2.$



Therefore the L- Shaped region has area $3\,ln\,3 - 2\,ln\,2.$

The area between the curve, the y-axis, $ln\,2$ and $ln\,3$ is given by the integral

$$\int_{ln\,2}^{ln\,3} e^y\,dy = \left[e^y\right]_{ln\,2}^{ln\,3}$$

$$= 3 - 2$$

$$= 1$$

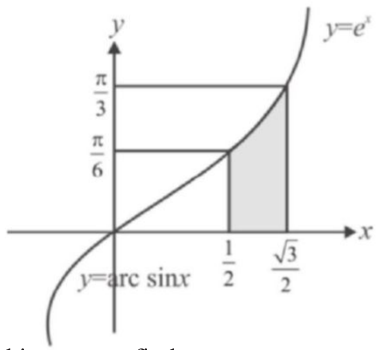
Putting all these together means the area represented by $ln\,x\,dx$ can be found by

$$\int_2^3 ln\,x\,dx = 3\,ln\,3 - 2\,ln\,2 - \int_{ln\,2}^{ln\,3} e^y\,dy$$

$$= 3\,ln\,3 - 2\,ln\,2 - 1$$

The answer can be written in several different forms. If we combined the logarithms we will ended up with ln $ln\left(\frac{27}{4}\right) - 1$ or $ln\left(\frac{27}{4e}\right)$.

Using a similar method we can find $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \arcsin x\,dx$



In this case we find

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \arcsin x\,dx = \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{6}\cdot\frac{1}{2} - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin y\,dy$$

$$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{12} + \frac{1-\sqrt{3}}{2}$$

We have seen that we can find the definite integral of any function if it has an inverse function that is easy to integrate.

Formula

Suppose the function f is one-to-one and increasing. Then, a geometric equivalence may be established:

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a).$$

Suppose the function f is one-to-one and decreasing. Then, another geometric equivalence may be established:

$$\int_a^b f(x)dx - \int_{f(b)}^{f(a)} f^{-1}(x)dx = (b-a)f(b) - a(f(a)-f(b)).$$

Integral of inverse functions

Inverse function integration is an indefinite integration technique. While simple, it is an interesting application of integration by parts. If f and f^{-1} are inverses of each other on some closed interval, then integral

$$f(x)dx = xf(x) - \int f^{-1}(f(x))f'(x)dx,$$

so integral $f(x)dx = xf(x) - G(f(x))$, where

$$G(x) = \int f^{-1}(x)dx.$$

Therefore, if it is possible to find an inverse f^{-1} of f, integrate f^{-1}, make the replacement x to $f(x)$, and subtract the result from $x f(x)$ to obtain the result for the original integral integral $f(x)dx$.

Examples

(1) Assume that $f(x) = exp(x)$, hence

$$f^{-1}(y) = ln(y).$$

The formula above gives immediately

$$\int ln(y)dy = y\,ln(y) - y + C.$$

(2) Similarly, with $f(x) = \cos(x)$ and

$$f^{-1}(y) = \arccos(y),$$

$$\int \arccos(y)dy = y\arccos(y) - \sin(\arccos(y)) + C.$$

(3) with $f(x) = \tan(x)$ and $f^{-1}(y) = \arctan(y)$,

$$\int \arctan(y)dy = y\arctan(y) + ln\left|\cos(\arctan(y))\right| + C.$$

Exercise

1. Let $f(x)$ be a one-to-one continuous function such that $f(1)=4$ and $f(6)=2$, and assume $\int_1^6 f(x)dx = 15.$ Calculate $\int_2^4 f^{-1}(x)dx.$
2. Evaluate

$$\int_1^7 ln(1+x)dx.$$

3. Let a function $f: R \to R$ be defined as $f(x) = x + \sin x$. The value of $\int_0^{2\pi} f^{-1}(x)dx$ will be

(a) $2\pi^2$ (b) $2\pi^2 - 2$ (c) $2\pi^2 + 2$ (d) π^2

HINTS & SOLUTIONS

1.Sol: The region bounded by f, $x=1$, and $y=2$ must have area 5, implying the integral in question corresponds to the area $5+1.\ (4-2)=7.$ The above formula for decreasing functions provides the same answer.

2.Sol: $8\,ln 8 - 2\,ln 2 - 6 = 22\,ln 2 - 6$

3.Sol: Using above formula, we get

$$\int_0^{2\pi}(x+\sin x)dx + \int_0^{2\pi} f^{-1}(x)dx = 2\pi(2\pi) - 0(f(0))$$

$$\Rightarrow \quad \left[\frac{x^2}{2} + \cos x\right]_0^{2\pi} + \int_0^{2\pi} f^{-1}(x)dx = 4\pi^2$$

i.e.,

$$\int_0^{2\pi} f^{-1}(x)dx = 4\pi^2 - \frac{4\pi^2}{2} - 1 + 0 + 1$$

$$= 2\pi^2$$

MATHEMATICS EXCEL

A Competitve Edge for JEE MAIN & ADVANCED

MANIPULATIONS OF TRIGONOMETRIC EXPRESSIONS

1. Evaluate $256\sin 10°\sin 30°\sin 50°\sin 70°$.

2. Let $a_1, a_2, \ldots, a_n$ be the sequence of all irreducible proper fractions with the denominator 24, arranged in ascending order. Find the value of

$$\sum_{i=1}^{n}\cos\left(a_i\pi\right).$$

3. Prove that

$$\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+n\beta)=$$

$$=\frac{\sin\frac{n\beta}{2}\cos(\alpha+\frac{n+1}{2}\beta)}{\sin\frac{1}{2}\beta}$$

4. Find the value of $\left(\cot 25°-1\right)\left(\cot 24°-1\right)$ $\left(\cot 23°-1\right)\ldots\left(\cot 20°-1\right)$.

5. Prove $\tan^{n}15°+\cot^{n}15°$ must be an even positive integer for any positive integer n

6. Prove that for any positive integer n,

$$\tan\alpha\tan 2\alpha+\tan 3\alpha\tan 3a+\ldots$$

$$+\tan(n-1)\alpha\tan n\alpha=\frac{\tan n\alpha}{\tan\alpha}-n$$

where $\tan\alpha\neq 0$ and $\tan\kappa\alpha\neq\pm\infty$ for $\kappa=1,2,\ldots,n$

7. Given $0<\alpha<\pi, \pi<\beta<2\pi$. If the equality

$$\cos(x+\alpha)+\sin(x+\beta)+\sqrt{2}\cos x=0$$

holds for any $x\in\mathbb{R}$, find the value of α and β

8. Find the value of

$$\sin^2 1°+\sin^2 2°+\sin^2 3°+\ldots+\sin^2 360°.$$

9. Find the smallest positive integer n such that

$$\frac{1}{\sin 45°\sin 46°}+\frac{1}{\sin 47°\sin 48°}+\ldots$$

$$+\frac{1}{\sin 133°\sin 134°}=\frac{1}{\sin n°}$$

10. Given $\sin\alpha+\sin\beta=\frac{\sqrt{6}}{3}, \cos\alpha+\cos\beta=\frac{\sqrt{3}}{3}$,

find the value of $\cos^2\frac{\alpha-\beta}{2}$

11. Evaluate $\cos\frac{\pi}{15}-\cos\frac{2\pi}{15}-\cos\frac{4\pi}{15}+\cos\frac{7\pi}{15}$

12. Evaluate $\cos 36°-\cos 72°$.

13. If $\frac{\cos 100°}{1-4\sin 25°\cos 25°\cos 50°}=\tan x°$, find x.

14. Prove that $\frac{1}{\sin 1°\sin 2°}+\frac{1}{\sin 2°\sin 3°}+\cdots$

$$+\frac{1}{\sin 89°\sin 90°}=\frac{\cos 1°}{\sin 1}.$$

15. Prove that $(i)\ \tan\frac{\pi}{5}\tan\frac{2\pi}{5}=\sqrt{5};$

$(ii)\ \tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5}=10$.

HINTS & SOLUTIONS

1.Sol: $256\sin 10°\sin 30°\sin 50°\sin 70°$

$$= 256\cos 20°\cos 40°\cos 60°\cos 80°$$

$$= \frac{128\sin 20°\cos 20°\cos 40°\cos 80°}{\sin 20°}$$

$$= \frac{64\sin 40°\cos 40°\cos 80°}{\sin 20°}$$

$$= \frac{32\sin 80°\cos 80°}{\sin 20°}$$

$$= \frac{16\sin 160°}{\sin 20°} = 16.$$

2.Sol: All the irreducible proper fractions with denominator 24 are

$$\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}, \frac{13}{24}, \frac{17}{24}, \frac{19}{24}, \frac{23}{24}.$$

since $$\frac{1}{24}+\frac{23}{24}=\frac{5}{24}+\frac{19}{24}$$

$$=\frac{7}{24}+\frac{17}{24}=\frac{11}{24}+\frac{13}{24}=1 \text{ and}$$

$\cos\alpha+\cos(\pi-\alpha)=0$, it follows that

$$\sum_{i=1}^{n}\cos(a_i\pi)=0+0+0+0=0$$

3.Sol: For any $k = 1, 2, \cdots,$

$$\sin\frac{1}{2}\beta\cos(\alpha+k\beta)=\frac{1}{2}\left[\left(\sin\alpha+\frac{2k+1}{2}\beta\right)\right]$$

$$-\sin\left(\alpha+\frac{2k-1}{2}\beta\right)$$

implies that $\sum_{k=1}^{n}\sin\frac{1}{2}\beta.\cos(\alpha+k\beta)$

$$=\frac{1}{2}\sum_{k=1}^{n}[\sin(\alpha+\frac{2k+1}{2}\beta)-\sin(\alpha+\frac{2k-1}{2}\beta)]$$

$$=\frac{1}{2}\left[\sin\left(\alpha+\frac{2n+1}{2}\beta\right)-\sin\left(\alpha+\frac{1}{2}\beta\right)\right]$$

$$=\sin\frac{n\beta}{2}\cos\left(\alpha+\frac{n+1}{2}\beta\right)$$

4.Sol: When $\alpha+\beta=45°$, then

$$1=\tan 45°=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad 1\cdot\tan\alpha-\tan\beta=\tan\alpha\cdot\tan\beta$$

$$(\cot\alpha-1)(\cot\beta-1)=\frac{(1-\tan\alpha)(1-\tan\beta)}{\tan\alpha\tan\beta}$$

$$=\frac{2\tan\alpha\tan\beta}{\tan\alpha\tan\beta}=2.$$

Thus, $(\cot 25°-1)(\cot 24°-1)\cdots(\cot 20°-1)$

$$=[(\cot 25°-1)(\cot 20°-1)]\cdots[(\cot 23°-1)$$

$$(\cot 22°-1)]$$

$$=2^3=8.$$

5.Sol: $\tan 15°=\frac{\tan 60°-\tan 45°}{1+\tan 60°\tan 45°}=\frac{\sqrt{3}-1}{1+\sqrt{3}}=2-\sqrt{3},$

and

$$\cot 15°=\frac{1}{2-\sqrt{3}}=2+\sqrt{3}, \tan^n 15°+\cot^n 15°$$

$$15°=(2-\sqrt{3})^n+(2+\sqrt{3})^n$$

using the binomial expansion,it follows that

$$(2-\sqrt{3})^n=2^n-\binom{n}{1}2^{n-1}\sqrt{3}+\binom{n}{2}2^{n-2}\left(\sqrt{3}\right)^2-\ldots$$

$$+(-\sqrt{3})^n$$

$$(2-\sqrt{3})^n=2^n+\binom{n}{1}2^{n-1}\sqrt{3}+\binom{n}{2}2^{n-2}(\sqrt{3})^2-\ldots$$

$$+(\sqrt{3})^n$$

Therefore in the sum $(2-\sqrt{3})^n+(2-\sqrt{3})^n$

Only the terms with even powers of $\sqrt{3}$ appear, and each of them appeared in pair, so the sum is an even positive integer.

6.Sol: The formula

$$\tan(k+1)\alpha=\frac{\tan k\alpha+\tan\alpha}{1-\tan\alpha\tan k\alpha} \text{ gives}$$

$$\tan k\alpha\tan(k+1)\alpha=\left(\frac{\tan(k+1)\alpha}{\tan\alpha}-\frac{\tan k\alpha}{\tan\alpha}\right)-1$$

$$\sum_{k=1}^{n-1}\tan k\alpha\tan(k+1)\alpha=\sum_{k=1}^{n-1}\left[\left(\frac{\tan(k+1)\alpha}{\tan\alpha}-\frac{\tan k\alpha}{\tan\alpha}\right)-(n-1)\right]$$

$$=\frac{\tan k\alpha}{\tan\alpha}-n.$$

7.Sol: Write the given equality in the form

$$(\cos\alpha+\sin\beta+\sqrt{2})\cos x+(\cos\beta-\sin\alpha)\sin x=0,$$

which holds for any real x, so

$$\begin{cases}\cos\alpha+\sin\beta+\sqrt{2})=0\\ \cos\beta-\sin\alpha=0,\end{cases}\quad\text{or}$$

$$\begin{cases}\sin\beta=-\cos\alpha-\sqrt{2}\\ \cos\beta=\sin\alpha.\end{cases}$$

By taking squares to both sides of each equality and add up them, then

$$(-\cos\alpha-\sqrt{2})^2+\sin^2\alpha=1$$

which given the solution $\cos\alpha=-\frac{1}{\sqrt{2}}$

Further, $0<\alpha<\pi$ implies that

$$\alpha=\frac{3\pi}{4},$$

so $\cos\beta=\sin\alpha=\frac{1}{\sqrt{2}}$.

Thus, $\beta=\frac{7\pi}{4}$ since $\pi<\beta<2\pi$

8.Sol: By using the formula in Q3 of (for $\alpha=\beta=2, n=180$),

$$\sin^2 1°+\sin^2 2°+\sin^2 3°+\cdots+\sin^2 360°$$

$$=2(\sin^2 1+\sin^2 2°+\sin^2 3°+\cdots+\sin^2 180°)$$

$$=180-(\cos 2°+\cos 4°+\cos 6°+\cdots+\cos 360°)$$

$$=180-\sin 180°+\cos 181°/+\sin 1°=180.$$

9.Sol: Note that

$$\sin^2 1°+\sin[(x+1)°-x°]=\sin(x+1)°\cos x°-\cos(x+1)°\sin x°$$

$$\frac{\sin 1°}{\sin x°\sin(x+1)°}$$

$$=\frac{\cos x°\sin(x+1)°-\sin x°\cos(x+1)°}{\sin x°\sin(x+1)°}$$

$$=\cot x°-\cot(x+1)°.$$

Multiplying both sides of the given equation by $\sin 1°$, we have

$$\frac{\sin 1°}{\sin n°}=(\cot 45°-\cot 46°)+\cot 47°-\cot 48°)+\cdots+(\cot 133°-\cot 134°)$$

$$=\cot 45°-(\cot 46°+\cot 134°)+(\cot 47°+\cot 133°)-\cdots+(\cot 89°+\cot 91°)-\cot 90°$$

$$=1.$$

Therefore, $\sin n°=\sin 1°$, and the least possible integer value for n is 1.

10.Sol:

$(\sin\alpha+\sin\beta)^2=\frac{2}{3}$, $(\cos\alpha+\cos\beta)^2=\frac{1}{3}$. By adding up them, it is obtained that

$$2+2(\sin\alpha.\sin\beta+\cos\alpha.\cos\beta)=1,$$

so $1+\cos(\alpha-\beta)=\frac{1}{2}$, hence $\cos^2\frac{\alpha-\beta}{2}=\frac{1}{4}$.

11.Sol: From the formulas for changing sum or difference to product,

$$\cos\frac{\pi}{15}-\cos\frac{2\pi}{15}-\cos\frac{4\pi}{15}+\cos\frac{7\pi}{15}$$

$$=\left(\cos\frac{\pi}{15}+\cos\frac{7\pi}{15}\right)-\left(\cos\frac{2\pi}{15}+\cos\frac{4\pi}{15}\right)$$

$$=2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}-\cos\frac{\pi}{5}\cos\frac{\pi}{15}$$

$$=2\cos\frac{\pi}{5}\left(\cos\frac{4\pi}{15}-\cos\frac{\pi}{15}\right)$$

$$=-4\cos\frac{\pi}{5}\sin\frac{\pi}{10}\sin\frac{\pi}{6}$$

$$=-2\cos\frac{\pi}{5}\sin\frac{\pi}{10}=-\frac{1}{2}$$

since $\cos 180°=\sin 72°=2\sin 36°\cos 36°$ $=4\sin 18°\cos 18°\cos 36°$ implies that $1=4\sin 18°\cos 36°$

$$\therefore\quad 2\sin 18°\cos 36°=\frac{1}{2}$$

12.Sol: Note that $\cos 36°-\cos 72°$

$$= \frac{2(\cos 36° - \cos 72°)(\cos 36° + \cos 72°)}{2(\cos 36° + \cos 72°)}$$

$$= \frac{2\cos^2 36° - 2\cos^2 72°}{2(\cos 36° + \cos 72°)}$$

By the double -angle formulas, the above equality becomes $\cos 36° - \cos 72°$

$$= \frac{\cos 72° + 1 - \cos 144° - 1}{2(\cos 36° + \cos 72°)}$$

$$= \frac{\cos 72° + \cos 36°}{2(\cos 36° + \cos 72°)} = \frac{1}{2}$$

13.Sol: By using the double angle formulas and the half angle formulas,

$$\frac{\cos 100°}{1 - 4\sin 25° \cos 25° \cos 50°}$$

$$= \frac{\cos 100°}{1 - 2\sin 50 \cos 50°}$$

$$= \frac{\cos^2 50° - \sin^2 50°}{(\cos 50° - \sin 50°)^2}$$

$$= \frac{\cos 50° + \sin 50°}{\cos 50° - \sin 50°}$$

$$= \frac{1 + \tan 50°}{1 - \tan 50°}$$

$$= \frac{\tan 45° + \tan 50°}{1 - \tan 45° \tan 50°} - \tan 95°, \therefore x = 95.$$

14.Sol: The left-hand side of the desired equation equal to

$$\sum_{k=1}^{89} \frac{1}{\sin k° \sin(k+1)°}$$

$$= \sum_{k=1}^{89} [\cot k° - \cot(k+1)°]$$

$$= \frac{1}{\sin 1°}.\cot 1° = \frac{\cos 1°}{\sin^2 1°},$$

15.Sol: We construct an equation with roots

$\tan\frac{n\pi}{5}, n = 0,1,2,3,4$ as follows. Since the equation $\tan 5\theta = 0$ for $\theta \in [0, \pi)$ has roots $\frac{n\pi}{5}, n = 0,1,2,3,4$ then each of the five roots satisfies the equation $\tan 3\theta = -\tan 2\theta$, therefore, by the muliple angle formulae,it satisfies the equation

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{-2\tan\theta}{1 - \tan^2\theta}$$

Letting $x = \tan\theta$, we have $\frac{3x - x^3}{1 - 3x^2} = \frac{-2x}{1 - x^2}$,or equivalently,

$$x(x^4 - 10x^2 + 5) = 0 \qquad (1)$$

If consider non -zero roots , then it becomes

$$x^4 - 10x^2 + 5 = 0.$$

Thus, $\tan\theta$ for $\theta = \frac{n\pi}{5}, n = 0,1,2,3,4$ are the four roots of (1).By the viete's theorem,

$$\tan\frac{\pi}{5}.\tan\frac{2\pi}{5}.\tan\frac{3\pi}{5}.\tan\frac{4\pi}{5} = 5 \qquad (2)$$

$$\tan\frac{\pi}{5}.\tan\frac{2\pi}{5} + \tan\frac{\pi}{5}.\tan\frac{3\pi}{5} + \tan\frac{\pi}{5}.\tan\frac{4\pi}{5}$$
$$+ \tan\frac{2\pi}{5}.\tan\frac{3\pi}{5} + \tan\frac{2\pi}{5}.\tan\frac{4\pi}{5} + \tan\frac{3\pi}{5}.\tan\frac{4\pi}{5}$$
$$= -10$$

Since $\tan\frac{\pi}{5} > 0, \tan\frac{2\pi}{5} > 0$ and

$$\tan\frac{3\pi}{5} = -\tan\frac{2\pi}{5}, \tan\frac{4\pi}{5} = -\tan\frac{\pi}{5},$$

(2) gives

$$\tan^2\frac{\pi}{5}.0, \tan^2\frac{2\pi}{5} = 5,$$

$\therefore \tan\frac{\pi}{5}.\tan\frac{2\pi}{5} = \sqrt{5}$, (i) is proven.

From (2),

$$\tan\frac{\pi}{5}.\tan\frac{2\pi}{5} - \tan\frac{\pi}{5}.\tan\frac{2\pi}{5} - \tan^2\frac{\pi}{5} - \tan^2\frac{2\pi}{5}$$
$$- \tan\frac{2\pi}{5}.\tan\frac{\pi}{5} + \tan\frac{2\pi}{5}.\tan\frac{\pi}{5} = -10,$$

$\therefore \tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$, (ii) is proven.

MOCK TEST PAPER

JEE MAIN - 5

2018 2019

1. If $A=\{1,2,3\}$, $B=\{x,y\}$ then $A\times B=$

(a) $\{1,2,3,x,y\}$

(b) $\{(1,x),(2,y),(1,y)\}$

(c) $\{(x,a),(y,b),(x,3)\}$

(d) $\{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$

2. *f* and *g* are two functions such that $(fg)(x)=(gf)(x)$ for all x. Then *f* and *g* may be defined as

(a) $f(x)=x^3, g(x)=x+1$

(b) $f(x)=x^m, g(x)=x^n$ where m, n are unequal integers

(c) $f(x)=\sqrt{x}, g(x)=\cos x$

(d) $f(x)=x-1, g(x)=x^2+1$

3. Set of points of discontinuity of $\frac{x^2}{[x]}$ is

(a) {0} (b) R (c) R^+ (d) Z

4. The function $f(x)=\frac{x}{\log_e x}$ increases on the interval

(a) $(0,e]$ (b) $(0,\infty)$

(c) $[e,\infty)$ (d) None of these

5. Let *I* be an open interval contained in the domain of a real function '*f*', then $f(x)$ is called strictly decreasing function in *I* if

(a) $x_1<x_2$ in $I\Rightarrow f(x_1)<f(x_2)$

(b) $x_1<x_2$ in $I\Rightarrow f(x_1)>f(x_2)$

(c) $x_1=x_2$ in $I\Rightarrow f(x_1)=f(x_2)$

(d) $x_1=x_2$ in $I\Rightarrow f(x_1)<f(x_2)$

6. $\int\frac{dx}{\sin(x-a)\sin(x-b)}$ is:

(a) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+c$

(b) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+c$

(c) $\log\sin(x-a)\sin(x-b)+c$

(d) $\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|$

7. $\int_0^{\pi}\frac{x}{1+\cos^2 x}dx=$

(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^2}{\sqrt{2}}$ (c) $\frac{\pi^2}{2\sqrt{2}}$ (d) $\frac{\pi^2}{4}$

8. Area of the region enclosed between the curves $x=y^2-1$ and $x=|y|\sqrt{1-y^2}$ is

(a) 1 (b) 4/3 (c) 2/3 (d) 2

9. The equation of line, which bisect the line joining two points (2, -19) and (6, 1) and perpendicular to the line joining two points (-1, 3) and (5, -1) is

(a) $3x-2y=30$ (b) $2x-y-3=0$

(c) $2x+3y=20$ (d) None of these

10. A line passing through the point $P(4,2)$, meets the x-axis and y-axis at A and B respectively. If O is the origin, then locus of the center of the circum circle of ΔOAB is

(a) $x^{-1}+y^{-1}=2$ (b) $2x^{-1}+y^{-1}=1$

(c) $x^{-1}+2y^{-1}=1$ (d) $2x^{-1}+2y^{-1}=1$

11. If the circles $x^2+y^2+2x+2ky+6=0$ and $x^2+y^2+2ky+k=0$ intersect orthogonally, then k is equal to

(a) 2 or $-\frac{3}{2}$ (b) -2 or $-\frac{3}{2}$

(c) 2 or $\frac{3}{2}$ (d) -2 or $\frac{3}{2}$

12. The equation of the hyperbola whose vertices are at (5, 0) and $(-5,0)$ and one of the directrices is $x=\frac{25}{7}$, is

(a) $\frac{x^2}{25}-\frac{y^2}{24}=1$ (b) $\frac{x^2}{24}-\frac{y^2}{25}=1$

(c) $\frac{x^2}{16}-\frac{y^2}{25}=1$ (d) $\frac{x^2}{25}-\frac{y^2}{16}=1$

13. The locus of point of intersection of perpendicular tangents of ellipse $\frac{(x-1)^2}{16}+\frac{(y-1)^2}{9}=1$ is :

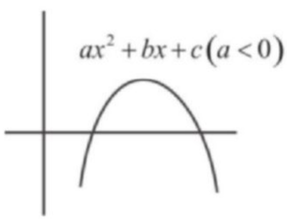
(a) $x^2+y^2=25$

(b) $x^2+y^2+2x+2y-23=0$

(c) $x^2+y^2-2x-2y-23=0$

(d) None of these

14. The diagram shows the graph of $y=ax^2+bx+c$. Then,



(a) $a>0$ (b) $b<0$

(c) $c<0$ (d) $b^2-4ac=0$

15. The value of m for which the equatios $x^3-mx^2+3x-2=0$ has two roots in same magnitude but opposite in sign is

(a) $\frac{4}{5}$ (b) $\frac{3}{4}$ (c) $\frac{2}{3}$ (d) $\frac{1}{2}$

16. If A and G be A.M. and G.M. of two given positive real numbers a and b respectively, then A and G are related as

(a) $A\geq G$ (b) $G\geq A$

(c) $A=G$ (d) $A=-G$

17. If $a,b,$ and c are in A.P., p and p' are, respectively, A.M. and G.M. between a and b while q,q' are, respectively, the A.M. and G.M. between b and c, then

(a) $p^2+q^2=p'^2+q'^2$ (b) $pq=p'q'$

(c) $p^2-q^2=p'^2-q'^2$ (d) None of these

18. If z_1 and z_2 are complex numbers satisfying $\left|\frac{z_1+z_2}{z_1-z_2}\right|=1$ and $\arg\left(\frac{z_1-z_2}{z_1+z_2}\right)=\pm n\pi\ (n\in\mathbb{Z})$

then $\frac{z_1}{z_2}$ is always

(a) Zero
(b) a rational number
(c) a positive real number
(d) a purely imaginary number

19. The coefficient of x^5 in the expansion of $(2-x+3x^2)^6$ is

(a) –4692 (b) –4694 (c) –4682 (d) 4592

20. A six-faced unbiased die is thrown twice and the sum of the numbers appearing on the upper face is observed to be 7. The probability that the number 3 has appeared atleast once, is

(a) $\frac{1}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

21. If $|\vec{a}|=11, |\vec{a}+\vec{b}|=30$ and $|\vec{a}-\vec{b}|=20$, then $|\vec{b}|=$

(a) 11 (b) 41 (c) 23 (d) 19

22. If $A(1,2,-1)$ and $B(-1,0,1)$ are given, then the coordinates of P which divides AB externally in the ratio 1 : 2 are

(a) $\frac{1}{3}(1,4,-1)$ (b) $(3,4,-3)$

(c) $\frac{1}{3}(3,4,-3)$ (d) None of these

23. The pairs of rectangular coordinate planes have equations

(a) $xy = yz = zx = 0$ (b) $x = y = z = 0$

(c) $xyz = 0$ (d) None of these

24. If the sides of a triangle are 4 cm, 5 cm, 6 cm then ratio of the least and greatest angle is

(a) 1:2 (b) 2 : 1 (c) 3 : 5 (d) 5 : 6

25. Number of real solutions of the equation

$\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$ where $-\pi \le x \le \pi$

(a) 0 (b) 1 (c) 2 (d) 4

26. If $2\cos^2 x + 47\cos x = 20\sin^2 x$, then what is the value of $\cos x$?

(a) $\frac{4}{11}$ (b) $\frac{-5}{2}$ (c) $\frac{-4}{11}$ (d) $\frac{2}{11}$

27. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$ then the values of k, a, b are respectively

(a) -6,4,9 (b) -6,12,18

(c) -6,-4,-9 (d) -6,-12,-12

28. If $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$, then $a^{-1}+b^{-1}+c^{-1}$ is equal to

(a) abc (b) -1

(c) 1 (d) None of these

29. $(\sim p \vee \sim q)$ is logically equivalent to-

(a) $p \wedge q$ (b) $\sim p \to q$

(c) $p \to \sim q$ (d) $\sim p \to \sim q$

30. The median of the numbers 6, 14, 12, 8, 10, 9, 11, is :-

(a) 8 (b) 10 (c) 10.5 (d) 11

ANSWER KEY

1. d	2. b	3. a	4. c	5. b
6. a	7. c	8. d	9. a	10. b
11. a	12. a	13. c	14. b	15. c
16. a	17. c	18. d	19. a	20. c
21. c	22. b	23. a	24. a	25. c
26. d	27. c	28. b	29. c	30. b

HINTS & SOLUTIONS

1.Sol: $A \times B = \{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$

2.Sol: $(fg)(x) = f(g(x)) = f(x^n) = (x^n)^m = x^{nm}$

$$(gf)x = g(f(x)) = g(x^m) = (x^m)^n = x^{mn}$$

3.Sol: Clearly the given function cannot be defined at $x = 0$

4. Sol: Given $f(x) = \dfrac{x}{\log e^x}$

$$\Rightarrow \quad x > 0$$

Now $\quad f'(x) = \dfrac{\log x - 1}{(\log x)^2}$

and also given $f(x)$ is increasing

i.e., $f'(x) \ge 0$

$$\Rightarrow \frac{\log x - 1}{(\log x)^2} \ge 0$$

i.e., $\log x \ge 1$

$$\Rightarrow \quad x \ge e$$

5.Sol: Conceptual

6.Sol: Method 1:

Let $I = \displaystyle\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]}{\sin(x-a)\sin(x-b)} dx = \frac{1}{\sin(b-a)}$$

$$\int \frac{\{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)\}}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int\{\cot(x-b)-\cot(x-a)\}dx$$

$$=\frac{1}{\sin(b-a)}\left[\log\sin(x-b)-\log\sin(x-a)\right]+c$$

$$=\frac{-1}{\sin(a-b)}\left[\log\frac{\sin(x-b)}{\sin(x-a)}\right]+c$$

Method 2 :

This problem can be solved using trigonometry, but Iam presenting here using complex number is let $z=e^{ix},\alpha=e^{ia}$ and $\beta=e^{ib}$, we have

$$\int\frac{1}{\sin(x-a)\sin(x-b)}dx$$

$$=\int\frac{1}{\left(\frac{za^{-1}-az^{-1}}{2i}\right)\left(\frac{z\beta^{-1}-\beta z^{-1}}{2i}\right)}\frac{dz}{zi}$$

$$=4i\alpha\beta\int\frac{zdz}{\left(z^2-\alpha^2\right)\left(z^2-\beta^2\right)}$$

$$=\frac{4i\alpha\beta}{\alpha^2-\beta^2}\left[\frac{z}{z^2-\alpha^2}-\frac{z}{z^2-\beta^2}\right]dz$$

$$=\frac{2i}{\alpha\beta^{-1}-\beta\alpha^{-1}}\left(\log\left(z^2-\alpha^2\right)-\log\left(z^2-\beta^2\right)\right)+c$$

$$=\frac{1}{\sin(a-b)}\log\left(\frac{z\alpha^{-1}-\alpha z^{-1}}{z\beta^{-1}-\beta z^{-1}}\right)+c$$

$$=\frac{1}{\sin(a-b)}\log\left[\frac{\sin(x-a)}{\sin(x-b)}\right]$$

7.Sol: Let $I=\int_0^{\pi}\frac{x}{1+\cos^2 x}dx \quad (1)$

$$\Rightarrow \quad I=\int_0^{\pi}\frac{(\pi-x)}{1+\cos^2(\pi-x)}dx \quad (2)$$

Adding (1) and (2), we get :

$$\Rightarrow \quad 2I=\int_0^{\pi}\frac{\pi}{1+\cos^2 x}dx$$

$$\Rightarrow \quad I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{1+\cos^2 x}=\frac{2\pi}{2}\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}$$

$$\Rightarrow \quad I=\pi\int_0^{\pi/2}\frac{\sec^2 x}{\sec^2+1}dx$$

Put $\tan x=t\Rightarrow\sec^2 xdx=dt$

For $x=\pi/2, t\to\infty$ and for $x=0, t=0$

$$\Rightarrow \quad I=\pi\int_0^{\infty}\frac{dt}{2+t^2}$$

$$\Rightarrow \quad I=\left|\frac{\pi}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right|_0^{\infty}=\frac{\pi}{\sqrt{2}}\times\frac{\pi}{2}=\frac{\pi^2}{2\sqrt{2}}$$

8.Sol: Given curves are $x=y^2-1$ and $x=|y|\sqrt{1-y^2}$

$\therefore$ The required area m.

$$A=2\int_0^1\left[y\sqrt{1-y^2}-\left(y^2-1\right)\right]dy$$

$$=\frac{-2}{3}\left(1-y^2\right)^{3/2}\Big|_0^1-\left(\frac{2y^3}{3}-2y\right)_0^1=2$$

9.Sol: Let L be the desired line.

Given that, L bisects the line joining two points $A(2,-19)$ and $B(6, 1)$. That is midpoint of AB is $M(4,-9)$ and also given that L is perpendicular to the line (L_1) joining two points $(-1,3)$ and $(5,-1)$. That is product of slope of the line L and the L_1 is -1

Now, The required equation of the line L is

$$\Rightarrow \quad y+9=\frac{-1}{\left(\frac{-1-3}{5+1}\right)}(x-4)$$

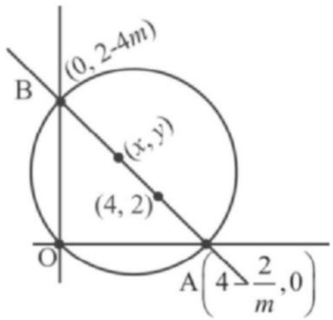
$$\Rightarrow \quad y+9=\frac{3}{2}(x-4)$$

i.e., $\quad 3x-2y-30=0$

10.Sol: Let the equations of the line be

$$y-2=m(x-4)$$

Then $A\left(4-\frac{2}{m},0\right)$ and $B(0,2-4m)$



now $(x, y)=\left(\frac{4-\frac{2}{m}+0}{2}, \frac{0+2-4m}{2}\right)$

$\Rightarrow \quad x=2-\frac{1}{m}$ and $y=1-2m$

i.e., $\quad x-2=\frac{-1}{m}, y-1=-2m$

eliminating m, we get

$(x-2)(y-1)=2$

$\Rightarrow \quad xy-2y-x+2=2$

$\Rightarrow \quad xy-2y-x=0$

i.e., $\quad x+2y=xy$

$\Rightarrow \quad \frac{1}{y}+\frac{2}{x}=1$

i.e., $\quad 2x^{-1}+y^{-1}=1$

11.Sol: Given that circle intersect orthogonally.

i.e., $\quad 2(g_1g_2+f_1f_2)=c_1+c_2$

$\Rightarrow \quad 2(1\cdot 0+k\cdot k)=6+k$

$\Rightarrow \quad 2k^2-k-6=0$

$\Rightarrow \quad (2k+3)(k-2)=0$

$\therefore \quad k=-\frac{3}{2}$ or 2

12.Sol: Given vertices are $(5,0),(-5,0)$.

$\Rightarrow \quad a=5$

Let one of the directrix, let $x=\frac{a}{e}$ given as

$x=\frac{25}{7}$

$\Rightarrow \quad e=\frac{7}{5}$

now $\quad b^2=a^2(e^2-1)=25\left(\frac{49}{25}-1\right)=24$

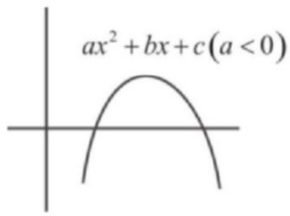
Equation of hyperbola is $\frac{x^2}{25}-\frac{y^2}{24}=1$.

13.Sol: Locus is director circle given by

$(x-1)^2+(y-1)^2=16+9$

$\Rightarrow \quad x^2+y^2-2x-2y-23=0$

14.Sol: As it is clear from the figure that it is a parabola opening downwards i.e. $a<0$.



Now, if $ax^2+bx+c=0 \Rightarrow$ it has two roots x_1 and x_2 as it cuts the axis distinct point x_1 and x_2. Now from the figure it is also clear that $x_1+x_2<0$ (i.e. sum of roots are negative)

$\Rightarrow \frac{-b}{a}<0 \Rightarrow \frac{b}{a}>0, b<0$

15.Sol: Let α, β be the three roots given $\alpha=-\beta$

i.e., $\quad \alpha+\beta=0$

Also $\quad \alpha+\beta+\gamma=m$

$\therefore \quad 0+\gamma=m$ i.e., $\gamma=m$

$\therefore \ \gamma$ is a root of given equation

$\therefore \quad \gamma^3-m\gamma^2+3\gamma-2=0$

i.e., $\quad m^3-m.m^2+3m-2=0$

$\therefore \quad m=\frac{2}{3}$

16.Sol: Let A and G be A.M. and G.M. of two given positive real numbers a and b respectively.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

This, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

Hence, $A - G \geq 0 \Rightarrow A \geq G$

17.Sol: $2b = a + c$

a, p, b, q, c are in A.P. Hence,

$$p = \frac{a+b}{2} \text{ and } q = \frac{b+c}{2}$$

Again, a, p', b, q', and c are in G.P. Hence,

$$p' = \sqrt{ab} \text{ and } q' = \sqrt{bc}$$

$$\Rightarrow p^2 - q^2 = \frac{(a-c)(a+c+2b)}{4}$$

$$= (a-c)b = ab - bc = (p')^2 - (q')^2$$

18.Sol: Given that $\left|\frac{z_1+z_2}{z_1-z_2}\right| = 1$

$$\Rightarrow \left|\frac{\frac{z_1}{z_2}+1}{\frac{z_1}{z_2}-1}\right| = 1$$

i.e., $\frac{\frac{z_1}{z_2}+1}{\frac{z_1}{z_2}-1} = \cos\alpha + i\sin\alpha$

$$\Rightarrow \frac{2\frac{z_1}{z_2}}{2} = \frac{1+\cos\alpha+i\sin\alpha}{\cos\alpha+i\sin\alpha-1}$$

i.e., $\frac{z_1}{z_2} = -i\cot\left(\frac{\alpha}{2}\right)(\alpha \pm n\pi)$

$\therefore$ $\frac{z_1}{z_2}$ is purely imaginary.

19.Sol: We have $\left(2-x+3x^2\right)^6 = \left[2-x(1-3x)\right]^6$

$$= \left[x(1-3x)-2\right]^6$$

Now $\left[x(1-3x)-2\right]^6 = \binom{6}{0}x^6(1-3x)^6 - \binom{6}{1}x^5(1-3x)^5\cdot 2$

$$+\binom{6}{2}x^4(1-3x)^4 2^2 + \binom{6}{3}x^3(1-3x)^3 2^3$$

$$+\binom{6}{4}x^2(1-3x)^2\cdot 2^4 - \ldots$$

Thus, the coeficient of x^5 in $\left(2-x+3x^2\right)^6$ is

$$-\binom{6}{1}\cdot 2 + \binom{6}{2}4\cdot(-3)\cdot 2^2 - \binom{6}{3}\cdot 3(-3)^3\cdot 2^3$$

$= -12 - 1440 - 3240 = -4692$

20.Sol: Sum of numbers appearing on the dice is 7.

S = {(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)}

i.e., $n(S) = 6$

Let E = Event of getting 3 atleast once

i.e., $n(E) = 2$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$

21.Sol: We know, $\left|\vec{a}-\vec{b}\right|^2 + \left|\vec{a}+\vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)$

$$\Rightarrow 400 + 900 = 2(121 + \left|\vec{b}\right|^2) \Rightarrow 650 - 121 = \left|\vec{b}\right|^2$$

$\therefore$ $\left|\vec{b}\right| = 23$

22.Sol: Given that $P(x, y, z)$ divides the line AB externally in the ratio 1 : 2.

i.e., $x = \frac{1(-1)-2(1)}{1-2}, y = \frac{1(0)-2(2)}{1-2}$, and

$$z = \frac{1(1)-2(-1)}{1-2}$$

$\Rightarrow$ $x = 3, y = 4,$ and $z = -3$

$\therefore$ $P(x, y, z) = (3, 4, -3)$

23.Sol: $xy = 0, yz = 0, zx = 0.$

24.Sol: We know, if three sides of triangle is known, then the law of cosine helps us to define the given triangle.
Let a = 4, b = 5, c = 6, and A, B, C, are respective angles of the opposite side of the given length.

now $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

i.e., $\cos A = \dfrac{5^2 + 6^2 - 4^2}{2 \cdot 5 \cdot 6} = \dfrac{3}{4}$ and

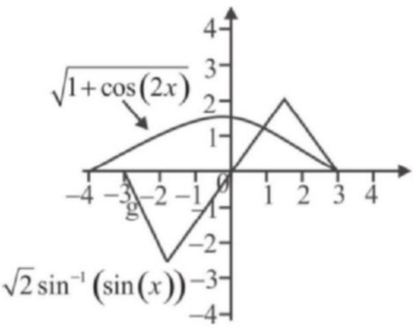
$$\cos C = \frac{4^2 + 5^2 - 6^2}{2 \cdot 4 \cdot 5} = \frac{1}{8}$$

$\Rightarrow \quad \lfloor A = 0.72$ and $\lfloor C = 1.44$

$\therefore \quad \lfloor A : \lfloor C = 0.72 : 1.44$

$= 1 : 2$

25.Sol: Clearly from the graph, we observe that the given equation has 2 real roots.



Aliter:

Given $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$

$$\Rightarrow \quad \sqrt{2}|\cos x| = \sqrt{2}\sin^{-1}(\sin x)$$

$$\therefore \quad |\cos x| = \sin^{-1}(\sin x)$$

$$\Rightarrow \quad \therefore x = \frac{4\pi}{17}, \frac{10\pi}{13}$$

26.Sol: Given $2\cos^2 x + 47\cos x = 20\sin^2 x$

$$\Rightarrow \quad 2\cos^2 x + 47\cos x = 20 - 20\cos^2 x$$

$$\Rightarrow \quad 22\cos^2 x + 47\cos x - 20 = 0$$

put $\cos x = t$

i.e., $22t^2 + 47t - 20 = 0$

27.Sol: $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

$$\Rightarrow k\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$\Rightarrow 2k = 3a, 3k = 2b, -4k = 24$$

$$\Rightarrow a = \frac{2k}{3}, b = \frac{3k}{2}, k = -6$$

28.Sol: Given $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$

$$\Rightarrow \quad (1+a)\{(1+b)(1+c)-1\}$$

$$-1(1+c-1) + 1(1-(1+b)) = 0$$

$$\Rightarrow (1+a)\{b+c+bc\} - c - b = 0$$

i.e., $bc + ab + ac + abc = 0$

$$\Rightarrow a^{-1} + b^{-1} + c^{-1} = -1$$

29.Sol:

p	q	~p	~q	~p ∨ ~q	p → ~q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

$\therefore \quad \sim p \vee \sim q$ is equivalent to $p \to \sim q$

30.Sol: Given observations are rewritten in ascending order, we get 6, 8, 9, 10, 11, 12, 14
$\therefore$ Median is 10.

QUICK RECAP

For competitive edge

SETS & RELATIONS

(1) The properties of inclusion

- $(\forall)A, A \subseteq A$ (reflexivity);
- $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$ (transitivity);
- $(\forall)A, \phi \subseteq A$
- If A is not part of the set B, then we write

  $A \not\subseteq B \Leftrightarrow (\exists)x(x \in A$ and $x \neq B)$.
- We will say that the **Set A is equal to the set B,** in short $A = B$, if they have exactly the same elements , that is

  $A = B \Leftrightarrow (A \subseteq B$ and $B \subseteq A)$

(2) Operations with sets

(I)Intersection of sets

The intersection of two sets A and B is defined as the set of those elements which are in both A and B and is written as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

The commutative, associative and distributive laws hold for intersection of two sets i.e.,

- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

In other words, we can write intersection as follows:

$$A \cap B = \{x \in E \mid x \in A \wedge x \in B\}$$

i.e., $x \in A \wedge B \Leftrightarrow x \in A \wedge x \in B$

$x \notin A \wedge B \Leftrightarrow x \notin A \wedge x \notin B$

(II)Union of sets

The union of two sets A and B is defined as the set of all elements which are either in A or in B or in both. The union of two sets is written as $A \cup B$. In other words, we can write union as follows:

$$A \cup B = \{x \in E \mid x \in A \vee x \in B)$$

i.e. $x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$,

$x \notin A \cup B \Leftrightarrow (x \notin A$ and $x \notin B)$

(III)Difference of sets

The difference of two set A and B, taken in this order, is defined as the set of all those elements of A which are not in B and is denoted by $A - B$ i.e.,

$A - B = \{x : x \in A$ and $x \notin B\}$.

In other words, we can write difference as follows:

$$A \setminus B = \{x \in E \mid x \in A \wedge x \notin B\}$$

i.e., $x \in A \setminus B \Leftrightarrow (x \in A \wedge x \notin B)$,

$x \notin A \setminus B \Leftrightarrow (x \notin A \wedge x \in B)$

(IV)Complement of a set

Complement of a set A is defined as E – A where E is the universal set and is denoted by A^c or

A' i.e., A S $A^c = E - A$ or

$A^c = \{x : x \in E, x \notin A\}$.

Note: $\left(A^c\right)^c = A, E^c = \phi, A \cap A^c = \phi, A \cup A^c = E$.

The complement of a set. Let $A \in P(E)$. The difference $E \backslash A$ is a subset of E, denoted E - A and called "the complement of A relative to E", that is

$$E - A = E \setminus A = \{x \in E \mid x \notin A\}.$$

In other words ,

$$x \in (E - A) \Leftrightarrow x \notin A,$$

$$x \notin (E - A) \Leftrightarrow x \in A,$$

(3) Properties of operations with sets

- $A \cap A = A, A \cup A = A$ (Idempotent laws)
- $A \cap B = B \cap A, A \cup B = B \cup A$ (Commutative laws)
- $(A \cap B) \cap C = A \cap (B \cap C)$;
- $(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity laws)
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive laws)
- $A \cup (A \cap B) = A$;
- $A \cap (A \cup B) = A$ (Absorption laws)
- $E - (A \cup B) = (E - A) \cap (E - B)$;
- $E - (A \cap B) = (E - A) \cup (E - B)$ (Morgan's laws)

Two "privileged " sets of E are ϕ and E. For any $A \in P(E)$, we have :

$\phi \subseteq A \subseteq E$,

$A \cup \phi = A$, $\quad A \cap \phi = \phi$, $\quad E - \phi = E$,

$A \cup \phi = A$, $\quad A \cap \phi = \phi$, $\quad E - E = \phi$,

$A \cup (E - A) = E$, $A \cap (E - A) = \phi$,

$E - (E - A) = A$

(Principle of reciprocity).

Subsequently , we will use the notation

$(E - A) = \overline{A}$.

(I) Symmetric difference

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

Properties:Irrespective of what the sets A,B and C are, we have :

- $A \Delta A = \phi$;
- $A \Delta B = B \Delta A$ (Commutativity);
- $A \Delta \phi = \phi \Delta A = A$;
- $A \Delta (A \Delta B) = B$;
- $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ (Associativity);
- $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$;
- $A \Delta B = (A \cup B) \setminus (A \cap B)$

(II) Application

Let A be a finite set. The number of elements in A is denoted by $n(A)$. Let A and B be two finite sets. If A and B are two disjoint sets, then

$$n(A \cup B) = n(A) + n(B).$$

If A and B are not disjoint, then

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A) = n(A - B) + n(A \cup B)$
- $n(B) = n(B - A) + n(A \cup B)$

(4) Cartesian Product

Let $A, B \in P(E)$. The set

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

is called a **Cartesian product** of sets A and B.

(5) Euler-Wenn diagrams

We call **Euler Diagrams** (in India Wenn's Diagrams) the figures that are used to interpret sets (circles, squares, rectangles etc.) and visually illustrate some properties of operations with sets. We will use the Euler circles.

(6) Relations

Definition: Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a,b) \in R$. Then we say that a is related to b by the relation R and write it as aRb. If $(a,b) \in R$, we write it as aRb.

(I) Total number of relations:

Let A and B be two non empty finite sets consisting of m and n elements respectively.

Then $A \times B$ consists of mn ordered pairs. So, total number of subset of B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal

(II)Domain and range of relation: Let R be a relation from a set A to a set B.

Then set of all first components or abscissa of the ordered pairs belonging to R is called the range of R.

(III)Inverse relation:

Let A,B be two sets and let R be a relation from a set *A* to a set *B*. Then the inverse of R, denoted by R^{-1}, is a relation from a set A to a set B. Then the inverse of R, is denoted by R^{-1}, is *a* relation from B to A and is defined by $R^{-1} = \{(b,a):(a,b) \in R\}$

Clearly $(a,b) \in R \Leftrightarrow (b,a) \in R^{-1}$.

Also, Dom $(R) =$ Range of (R^{-1}) and

Range of $(R) = Dom(R^{-1})$

RELATIONS & FUNCTIONS

(1) Set and elements relations

In the case of a finite set *A* (say of *n* elements), there is a simple interpretation of a relation. We simply draw an *n* table, representing all the possible pairs (*x,y*), and we put a '*' in a cell when the corresponding pair belongs to the relation. For example, with the set $A = \{a,b,c\}$, we could have the following relation:

	a	b	c
a		*	
b			
c	*		*

In this case, the relation contains the pairs (a,b), (c,a), and (c,c).

In general, for every way you can put stars in the above table (including none at all), you get a relation on *A*.

We will first examine a few simpler problems.

(I) All relations

In general, for a set of *n* elements, there are n^2 squares in the table, and $2^{(n^2)}$ possible relations.

(II) Reflexive relations

A relation is reflexive if it contains all the pairs (*x,x*) for every *x* in *A*.

For a set n elements, you would have:

$2^{(n^2-n)} = 2^{(n(n-1))}$ possible reflexive relations.

(III) Irreflexive relations

A relation is irreflexive if it contains none of the pairs (*x,x*). This means that you must have no '*' on the main diagonal, and you are still free to do whatever you want with the other squares.

(IV) Symmetric relations

A relation is symmetric if, whenever it contains the pair (*x,y*), it also contains the pair (*y,x*). This means that the table must be symmetric with respect to the main diagonal.

To build a symmetric relation, we can freely choose all the squares on and above the diagonal.

There are $\dfrac{n(n+1)}{2}$ such squares, and two possibilities for each of them, so the number of symmetric relation is

$$2^{\frac{n(n+1)}{2}}$$

(V) Antisymmetric relations

The number of pairs of distinct elements is "*n* choose 2":

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

and, as there are three possibilities for each pair, we have $3^{\frac{n(n-1)}{2}}$ possibilities for off-diagonal elements.

The total number of antisymmetric relations is thus:

$$2^n \cdot 3^{\frac{n(n-1)}{2}}$$

(VI) Reflexive and antisymmetric

If you compare that with the antisymmetric case, the only difference is that you must have *' in all diagonal squares-you are no longer free to select them. You still have 3 possibilities for each of the $\frac{n(n-1)}{2}$ pairs of distinct elements (off-diagonal squares), and the total number is therefore:

$$3^{\frac{n(n-1)}{2}}$$

(2) Functions

Definition : A function f from a domain A to a codomain B, notated as $f: A \to B$ is a map that maps every element in the domain to exactly one element in the codomain.

Definition: A mapping is defined as a function $f: A \to B$ where rule is $y = f(x)$ such that

(1) $\forall x \in A$

(2) There exists a unique $y \in B$

(3) Domain, range and co-domain of a function

(I) Domain:

Definition: For a given function $f: A \to B$, $D = \{x : x \in A$ such that f is well defined$\}$

(A)Rule for determining domain of function

(i) Algebraic functions

- For Rational Functions, exclude the value of x, which makes the denominator of the function zero.
- Expression under the even root should be non-negative.

(ii)Logarithmic Functions: $\log_b a$ is defined for $a > 0, b > 0$ and $b \neq 1$

(iii) Exponential Function : a^x is defined for all real values of x, where $a > 0$.

Rules for solving problems on the domain of a function

- $(x-a)(x-b) > 0 \Rightarrow x < a$ or $x > b$, for $a < b$
- $(x-a)(x-b) < 0 \Rightarrow a < x < b$ for $a < b$
- $|x| < a \Rightarrow -a < x < a$
- $|x| > a \Rightarrow x < -a$ or $x > a$
- $\log_b a > k \Rightarrow \begin{cases} a > b^k, & if\ b > 1 \\ a < b^k, & if\ 0 < b < 1 \end{cases}$
- $\sqrt{x^2} = |x|$
- $\sqrt[n]{x^n} = |x|$, if n is even and $\sqrt[n]{x^n} = |x|$, if n is odd.

(II)Range:

Definition (Range): The image or range of a function $f: A \to B$ is the set of all $y \in B$ such that $y = f(x)$ for some $x \in A$.

(4) Methods of determining Range

- **Representing x in terms of y:** If $y = f(x)$. Try to express as $g(x) = y$, then the domain of $g(y)$ represents possible values of y, which is range of $f(x)$.
- Graphical Method : The set of y - coordinates of the graph of a function is the range.
- Using monotonicity : Many of the functions are monotonic increasing or monotonic decreasing. In case of monotonic continuous functions the minimum and maximum values lie at end points of domain. Some of the common functions which are increasing or decreasing in the interval where they are continuous is as under.

Monotonic increasing	Monotonic decreasing
$\log_a x, a > 1$	$\log_a x, 0 < a < 1$
e^x	e^{-x}
$\sin^{-1} x$	$\cos^{-1} x$
$\tan^{-1} x$	$\cot^{-1} x$
$\sec^{-1} x$	$\csc^{-1} x$

For monotonic increasing functions in $[a, b]$

(1) $f'(x) \geq 0$

(2) Range $[f(a), f(b)]$

Algebra of functions

- **Addition of Functions**

$$(f+g)(x) = f(x) + g(x)$$

- **Subtraction of Functions**

  $(f-g)(x)=f(x)-g(x)$

- **Multiplication of Functions**

  $(f\,.\,g)(x)=f(x)\,.\,g(x)$

- **Division of Functions**

  $\frac{f}{g}(x)=\frac{f(x)}{g(x)}$ where $g(x)\neq 0$

(5) Composite Functions

Definition (Composition of Functions)

If $f:A\to B$ and $g:B\to C$ be two functions, then we defined composite of f and g as

$$(g\ o\ f)(x)=g(f(x))$$

(6) Iterated Function Composition

If the range of a function is a subset of the domain of a function, then we can compose this function with itself. If so, we use $f^2(x)$ to denote *fof(x)*.

More generally ,we say that $f^n(x)$ is f compose with itself n times, i.e.

$$\underbrace{f\ o\ f\ o\ldots o\ f}_{n\text{ times}}$$

(7) Important Functions

Properties of Greatest Integer Function:

- $x-1<[x]\le x$ or $[x]\le x<[x]+1$
- $[x+n]=[x]+n$, where $n\in Z$
- $x=[x]+\{x\}$ where $\{x\}$ is the fractional part of x
- $[-x]=\begin{cases}-[x]-1 & if\ x\notin Z\\ -[x] & if\ x\in Z\end{cases}$
- $x-1<[x]\le x$
- $x+y\ge[x+y]\ge[x]+[y]$
- $\left[\frac{x}{n}\right]+\left[\frac{x+1}{n}\right]+\left[\frac{x+2}{n}\right]+\ldots+\left[\frac{x+n-1}{n}\right]=[x]$

  or $[x]+\left[x+\frac{1}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx]$

- $\left[\frac{x}{n}\right]=\left[\frac{[x]}{n}\right]$ if $x\in Z^+$
- Inequalities

  (a) $[\phi(x)]\le I\ \Rightarrow \phi(x)<I+1$

  (b) $[\phi(x)]\ge I\ \Rightarrow \phi(x)\ge I$

Fractional Part : Fractional part function of x denoted as $\{x\}$ and defined as $\{x\}=x-[x]$ and hence $0\le x<1$

$$i.e., \{x\}=\begin{cases}\vdots, & \\ x+2, & -2\le x<-1\\ x+1, & -1\le x<0\\ x, & 0\le x<1\\ x-1, & 1\le x<2\\ \vdots, & \end{cases}$$

Properties of Fractional Part

- $0\le\{x\}<1$ which generalizes to $0\le f(x)<1$
- $\{-x\}=\begin{cases}1-\{x\}, & x\notin Z\\ 0 & x\notin Z\end{cases}$

Transcendental Functions:

Trigonometric Functions

Function	Domain	Range
$y=\sin x$	R	[-1,1]
$y=\cos x$	R	[-1,1]
$y=\tan x$	R-$\left\{\left\{(2n+1)\frac{\pi}{2}\middle|n\in I\right\}\right\}$	R
$y=\cot x$	R-$\{n\pi\|n\in I\}$	R
$y=\sec x$	R-$\left\{\left\{(2n+1)\frac{\pi}{2}\middle|n\in I\right\}\right\}$	$(-\infty,-1]\cup[1,\infty)$
$y=\operatorname{cosec} x$	R-$\{n\pi\|n\in I\}$	$(-\infty,-1]\cup[1,\infty)$

Inverse Trigonometric Functions

Function	Domain	Range
$y=\sin^{-1}x$	$-1\le x\le 1$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\cos^{-1}x$	$-1\le x\le 1$	$[0,\pi]$
$y=\tan^{-1}x$	$-\infty<x<\infty$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$y=\cot^{-1}x$	$-\infty<x<\infty$	$(0,\pi)$
$y=\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
$y=\operatorname{cosec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$

TRIGONOMETRY

(1) Rotations and Reflections of Angles

$$\sin(\theta+90^o)=\frac{x}{r}=\cos\theta, \cos(\theta+90^o)=\frac{-y}{r}$$

$$=-\sin\theta, \tan(\theta+90^o)=\frac{x}{-y}=-\cot\theta$$

- $\sin(\theta+90^o)=\cos\theta$
- $\cos(\theta+90^o)=-\sin\theta$
- $\tan(\theta+90^o)=-\cot\theta$
- $\sin(\theta-90^o)=-\cos\theta$
- $\cos(\theta-90^o)=\sin\theta$
- $\tan(\theta-90^o)=-\cot\theta$
- $\sin(\theta\pm180^o)=-\sin\theta$
- $\cos(\theta\pm180^o)=-\cos\theta$
- $\tan(\theta\pm180^o)=\tan\theta$
- $\sin(-\theta)=-\sin\theta$
- $\cos(-\theta)=\cos\theta$
- $\tan(-\theta)=-\tan\theta$
- $\sin(90^o-\theta)=-\cos\theta$
- $\cos(90^o-\theta)=\sin\theta$
- $\tan(90^o-\theta)=-\cot\theta$
- $\sin(180^o-\theta)=\sin\theta$
- $\cos(180^o-\theta)=-\cos\theta$
- $\tan(180^o-\theta)=-\tan\theta$

Identities

(2) Basic Trigonometric Identities

- $\csc\theta=\frac{1}{\sin\theta}$ *when* $\sin\theta\neq0$
- $\sin\theta=\frac{1}{\csc\theta}$ *when* $\csc\neq0$
- $\cot\theta=\frac{1}{\tan\theta}$ when $\tan\theta$ is defined and not 0
- $\sin\theta=\frac{1}{\csc\theta}$ when csc θ is defined and not 0
- $\cos\theta=\frac{1}{\sec\theta}$ when sec θ is defined and not 0
- $\tan\theta=\frac{1}{\cot\theta}$ when $\cot\theta$ is defined and not 0
- $\tan\theta=\frac{\sin\theta}{\cos\theta}$ *when* $\cos\theta\neq0$
- $\cot\theta=\frac{\cos\theta}{\sin\theta}$ *when* $\theta sin\neq0$
- $\sec\theta=-\frac{\sin\theta}{\cos\theta}$ *when cos* $\neq0$
- $\cos^2\theta+\sin^2\theta=1$
- $\sin^2\theta=1-\cos^2\theta$
- $\cos^2\theta=1-\sin^2\theta$
- $\sin\theta=\pm\sqrt{1-\cos^2\theta}$
- $\cos\theta=\pm\sqrt{1-\sin^2\theta}$
- $1\leq\sin\theta\leq1$
- $-1\leq\cos\theta\leq1$
- $1+\tan^2\theta=\sec^2\theta$

  likewise, we get other identity
- $\cot^2\theta+1=\csc^2\theta$

(3)Sum and Difference Formulas

- $\sin(A+B)=\sin A\cos B+\cos A$ sin
- $\cos(A+B)=\cos A\cos B$ - $\sin A\sin B$
- $\sin(A$ - $B)=\sin A\cos B$ - $\cos A\sin B$
- $\cos(A+B)=\cos A\cos B+\sin A\sin B$
- $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$
- $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

(4) Double-Angle and Half-Angle Formulas

- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
- $\cos 2\theta = 2\cos^2\theta - 1$
- $\cos 2\theta = 1 - 2\sin^2\theta$
- $\sin^2\frac{1}{2}\theta = \dfrac{1-\cos\theta}{2}$
- $\cos^2\frac{1}{2}\theta = \dfrac{1+\cos\theta}{2}$
- $\tan^2\frac{1}{2}\theta = \dfrac{1-\cos\theta}{1+\cos\theta}$
- $\sin\frac{1}{2}\theta = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
- $\cos\frac{1}{2}\theta = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
- $\tan\frac{1}{2}\theta = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$
- $\tan\frac{1}{2}\theta = \dfrac{1-\cos\theta}{\sin\theta}$
- $\tan\frac{1}{2}\theta = \pm\dfrac{\sin\theta}{1+\cos\theta}$
- $\cot\frac{1}{2}\theta = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

(5) Other Identities

- $\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$
- $\cos A\sin B = \frac{1}{2}(\sin(A+B)-\sin(A-B))$
- $\cos A\cos B = \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$
- $-\sin A\sin B = \frac{1}{2}\left(\cos(A+B)-\cos(A-B)\right)$

We can go in the opposite direction with the *sum-to-product formulas:*

- $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
- $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$
- $\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
- $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

INVERSE TRIGONOMETRY

(1) Inverse Sine and Cosine

Defining $\sin^{-1}(x)$ and $\cos^{-1}(x)$

For x in the interval $[-1,1], \sin^{-1}(x)$ is the angle measure in the interval $[-\pi/2, \pi/2]$ whose sine value is x.

For x in the interval $[-1,1], \cos^{-1}(x)$ is the angle measure in the interval $[0,\pi]$ whose cosine value is x.

(2) Inverse Tangent and Cotangent

For any x, $\tan^{-1}(x)$ is the angle measure in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent value is x.

For any $x, \cot^{-1}(x)$ is the angle measure in the interval $(0,\pi)$ whose cotangent value is x.

(3) Inverse Secant and Cosecant

For x in $(-\infty,-1]$ or $[1,\infty), \sec^{-1}(x)$ is the angle measure in $[0,\pi/2)$ or $(\pi/2,\pi)$ whose secant value is x.

For x in $(-\infty,-1]$ or $[1,\infty), \csc^{-1}(x)$ is the angle

measure in $[-\pi/2, 0)$ or $(0, \pi/2]$ whose cosecant value is x.

(4) Summary of Inverse Functions

Definitions

- For x in the interval $[-1,1], \sin^{-1}(x)$ is the angle measure in the interval $[-\pi/2, \pi/2]$ whose sine value is x.
- For x in the interval $[-1,1], \cos^{-1}(x)$ is the angle measure in the interval $[0, \pi]$ whose cosine value is x.
- For any x, $\tan^{-1}(x)$ is the angle measure in the interval $(-\pi/2, \pi/2)$ whose tangent value is x.
- For any $x, \cot^{-1}(x)$ is the angle measure in the interval $(0, \pi)$ whose cotangent value is x.
- For x in $(-\infty, -1] \cup [1, \infty), \sec^{-1}(x)$ is the angle measure in $[0, \pi/2) \cup (\pi/2, \pi]$ whose secant value is x.
- For x in $(-\infty, -1] \cup [1, \infty), \csc^{-1}(x)$ is the angle measure in $[-\pi/2, 0) \cup (0, \pi/2]$ whose cosecant value is x.

Identities

In the next table we summarize the relationships between the trigonometric functions and their inverses, along with the intervals on which they hold.

$\sin^{-1}(\sin(x)) = x$ for x in the interval $[-\pi/2, \pi/2]$. $\sin(\sin^{-1}(x)) = x$ for x in the interval $[-1,1]$.	$\sec^{-1}(\sec(x)) = x$ for x in $[0, \pi/2) \cup (\pi/2, \pi)$ $\sec(\sec^{-1}(x)) = x$ for x in $(-\infty, -1] \cup [1, \infty)$
$\cos^{-1}(\cos(x)) = x$ for x in the interval $[0, \pi]$. $\cos(\cos^{-1}(x)) = x$ for x in the interval $[-1,1]$.	$\csc^{-1}(\csc(x)) = x$ for x in $[-\pi/2, 0) \cup (0, \pi/2]$ $\csc(\csc^{-1}(x)) = x$ for x in $(-\infty, -1] \cup [1, \infty)$
$\tan^{-1}(\tan(x)) = x$ for x in the interval $(-\pi/2, \pi/2)$ $\tan(\tan^{-1}(x)) = x$ for any x.	$\cot^{-1}(\cot(x)) = x$ for x in the interval $(0, \pi)$ $\cot(\cot^{-1}(x)) = x$ for any x.

Here are some phase changes that translate one inverse function to another

(5) Properties

(I) $\mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}(x), \quad \forall x \in [-1,1]$

(II) $\mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}(x), \forall x \in [-1,1]$

(III) $\mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1} x, \quad \forall x \in R$

(IV) $\mathrm{Cosec}^{-1}(-x) = -\mathrm{Cosec}^{-1} x,$

$\forall x \in (-\infty, -1] \cup [1, \infty)$

(V) $\mathrm{Sec}^{-1}(-x) = \pi - \mathrm{Sec}^{-1} x,$

$\forall x \in (-\infty, -1] \cup [1, \infty)$

(VI) $Cot^{-1}(-x) = \pi - Cot^{-1} x, \forall x \in R$

(VII) $\cos^{-1}(x) = \pi/2 - \sin^{-1}(x)$ for x in the interval $[-1,1]$.

(VIII) $\cos^{-1}(x) = \pi/2 - \tan^{-1}(x)$ for any x.

(XI) $\csc^{-1}(x) = \pi/2 - \sec^{-1}(x)$ for x in $(-\infty, -1]$ or $[1, \infty)$.

(6) Inverting x also gives interesting connections

(I) $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right), x \le -1, x \ge 1$

(II) $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right), x \ge 1, x \le -1$

(III)
$$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right); & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right); & x < 0 \end{cases}$$

(7) Composing functions with inverses

(I) $\sin(\cos^{-1} x) = \cos(\sin^{-1} x)$

$$= \sqrt{1-x^2}, -1 \le x \le 1$$

(II) $\tan(\cot^{-1} x) = \cot(\tan^{-1} x) = \dfrac{1}{x}, x \in \mathbb{R}, x \neq 0$

(III) $\csc(\sec^{-1} x) = \sec(\csc^{-1} x) = \dfrac{|x|}{\sqrt{x^2-1}}, |x| > 1$

Identities of Addition and Subtraction

(8) Property - A

(I) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$x \ge 0, y \ge 0 \,\&\, (x^2+y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right],$

$x \ge 0, y \ge 0 \,\&\, (x^2+y^2) > 1$

Note :

$(x^2+y^2) \le 1 \Rightarrow 0 \le \sin^{-1}(x) + \sin^{-1}(y) \le \dfrac{\pi}{2}$

$(x^2+y^2) > 1 \Rightarrow 0 \le \sin^{-1}(x) + \sin^{-1}(y) < \pi$

(II) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right],$

$x \ge, y \ge 0$

(III) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x \ge 0, y \ge 0 \,\&\, xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-y}\right); & x \ge 0, y \ge 0 \,\&\, xy > 1 \\ \dfrac{\pi}{2} & x \ge 0, y \ge 0 \,\&\, xy = 1 \end{cases}$$

Note : $xy < 1 \Rightarrow 0 < \tan^{-1} + \tan^{-1} y < \dfrac{\pi}{2};$

$xy > 1, \dfrac{\pi}{2} < \tan^{-1} + \tan^{-1} y < \pi$

(9) Property - B

(I) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right),$

$x \ge 0, y \ge 0$

(II) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2} \cdot \sqrt{1-y^2}), \\ \qquad x \ge 0, y \ge 0, x \le y \\ -\cos^{-1}(xy + \sqrt{1-x^2} \cdot \sqrt{1-y^2}), \\ \qquad x \ge 0, y \ge 0, x > y \end{cases}$$

(III) $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left[\dfrac{x-y}{1+xy}\right], x \ge 0, y \ge 0$

(10) Property - C

(I) $$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}(x) & if\ |x| \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}(x) & if\ x > \dfrac{1}{\sqrt{2}} \\ -(\pi + 2\sin^{-1}(x)) & if\ x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

(II) $$\cos^{-1}\left(2x^2-1\right) = \begin{cases} 2\cos^{-1}(x) & if\ 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}(x) & if\ -1 \le x < 0 \end{cases}$$

(III) $$\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}(x) & if\ |x| < 1 \\ \pi + 2\tan^{-1}(x) & if\ x < -1 \\ -(\pi - 2\tan^{-1}(x)) & if\ x > 1 \end{cases}$$

(IV) $$\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}(x) & if\ |x| \le 1 \\ \pi - 2\tan^{-1}(x) & if\ x > 1 \\ -(\pi - 2\tan^{-1}(x)) & if\ x < -1 \end{cases}$$

(V) $$\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}(x) & if\ x \ge 0 \\ -2\tan^{-1}(x) & if\ x < 0 \end{cases}$$

(11) Property - D

$$\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

if $x > 0, y > 0, z > 0 \,\&\, (xy+yz+zx) < 1$

(I) If $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi$

then $x + y + z = xyz$

(II) if the $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \dfrac{\pi}{2}$

$xy + yz + zx = 1$

(III) $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\pi$

(IV) $\tan^{-1}(1)+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{2}$

(V) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}; \textit{if } -1<x<1$

(VI) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

(VII) $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$

APPLICATIONS OF TRIGONOMETRY

For a general triangle, which may or may not have a right angle, we will again need three pieces of information. The four cases are:

Case 1: One side and Two angles

Case 2: Two sides and one opposite angle

Case 3: Two sides and the angle between them

Case 4: Three sides

Note that if we were given all three angles we could not determine the sides uniquely; by similarity an infinite number of triangles have the same angles.

General Triangles

(1) The Law of Sines

Theorem 1 (The law of Sine). If a triangle has sides of lengths a,b, and c opposite the angles A,B, and C, respectively, then

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad (1)$$

Note that by taking reciprocals, equation (1) can be written as

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \qquad (2)$$

and it can also be written as a collection of three equations:

$$\frac{a}{b}=\frac{\sin A}{\sin B},\ \frac{a}{c}=\frac{\sin A}{\sin C},\ \frac{b}{c}=\frac{\sin B}{\sin C} \qquad (3)$$

Another way of stating the Law of sines is : *The sides of a triangle are proportional to the sines of their opposite angles.*

(2) The Law of Cosines

Theorem 2 (Law of Cosines :). If a triangle has sides of lengths a, b, and c opposite the angles A,B, and C, respectively, then

$$a^2=b^2+c^2-2bc\cos A, \qquad (4)$$

$$b^2=c^2+a^2-2ca\cos B, \qquad (5)$$

$$c^2=a^2+b^2-2ab\cos C, \qquad (6)$$

(3) The Law of Tangents

Theorem 3 (Law of Tangents). If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \qquad (7)$$

$$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} \qquad (8)$$

$$\frac{c-a}{c+a}=\frac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)} \qquad (9)$$

$$\frac{b-a}{b+a}=\frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)} \qquad (10)$$

Mollweide's equations: For any triangle ΔABC,

$$\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C},\ \textit{and} \qquad (11)$$

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}, \qquad (12)$$

(4) The Area of a Triangle

Case 1. *Two sides and the included angle.*

Suppose that we have a triangle ΔABC, in which A can be either acute, a right angle, or obtuse, as in Figure 3. Assume that a, b, and c are known.

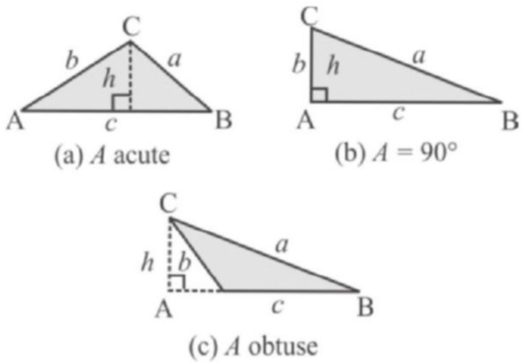


Fig-3: Area of ΔABC

In each case we draw an altitude of height h from the vertex at C to $\overline{AB}$, so that the area (which we will denote by the letter K) is given by $K = \frac{1}{2}hc$.

$$Area = K = \frac{1}{2}bc\sin A \qquad (13)$$

$$Area = K = \frac{1}{2}ac\sin B \qquad (14)$$

$$Area = K = \frac{1}{2}ab\sin C \qquad (15)$$

Case 2. *Three angles and any side*

$$Area = K = \frac{a^2 \sin B \sin C}{2\sin A} \qquad (16)$$

$$Area = K = \frac{b^2 \sin A \sin C}{2\sin B} \qquad (17)$$

$$Area = K = \frac{c^2 \sin A \sin B}{2\sin C} \qquad (18)$$

Case 3. *Three sides*

Heron's formula:

$$Area = K = \sqrt{s(s-a)(s-b)(s-c)} \qquad (19)$$

Heron's formula is rewritten as :

$$Area = K = \frac{1}{4}\sqrt{(a+(b+c))(c-(a-b))(c+(a-b))(a+(b-c))} \qquad (20)$$

$$A = K = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2} \qquad (21)$$

(5) Circumscribed and inscribed Circles

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Theorem 4. If an inscribed angle $\angle A$ and a central angle $\angle O$ intercept the same arc, then

$$\angle A = \frac{1}{2}\angle O.$$

Thus, inscribed angles which intercept the same are equal .

Theorem 5. For any triangle ΔABC , the radius R of its circumscribed circle is given by :

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (15)$$

Corollary 5.1. For any triangle , the centre of its circumscribed circle is the intersection of the perpendicular bisectors of the sides.

Theorem 6. For a triangle ΔABC , let K be its area and let R be the radius of its circumscribed circle. Then

$$k = \frac{abc}{4R} \text{ and hence } R = \frac{abc}{4K}. \qquad (16)$$

Corollary 5.2. For a triangle ΔABC ,

let $s = \frac{1}{2}(a+b+c)$. Then the radius R of its circumscribed circle is

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \qquad (17)$$

Theorem 7. For any triangle ΔABC, let $s = \frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r = (s-a)\tan\frac{1}{2}A = (s-b)\tan\frac{1}{2}B = (s-c)\tan\frac{1}{2}C \qquad (18)$$

Theorem 8. For any triangle ΔABC, let $s = \frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r = \frac{K}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad (19)$$

(I) Excircle, Excenter.

(1) The angle bisectors of $\angle A, \angle Z_1BC, \angle Y_1CB$ are all concurrent at I_1.

(2) I_1 is the center of the excircle which is the circle tangent to BC and to the extensions of AB and AC.

r_1 is the radius of the excircle.

(A) Properties:

(i) Elementary Length Formulae:

Theorem 9.

$AY = AZ = s-a,\ BZ = BX = s-b,$

$CX = CY = s-c.$

Theorem 10.

$BX_1 = BZ_1 = s-c,\ \ CY_1 = CX_1 = s-b,$

$AY_1 = AZ_1 = s.$

(B) Area Formulae:

Theorem 11.

$[ABC] = K = rs = r_1(s-a) = r_2(s-b)$

$= r_3(s-c)$

Where r_1, r_2 and r_3 are exradii

These are very useful when dealing with problems involving the inradius and the exradii. (Let R be the circumradius.)

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$r_1 + r_2 + r_3 - r = 4R$$

$$s^2 = r_1r_2 + r_2r_3 + r_3r_1.$$

$$[ABC] = \sqrt{rr_1r_2r_3}.$$

Here $[ABC]$ is the area of triangle.

(C) Radii RelationshipsComputing Lengths:

$$AI = r\operatorname{cosec}\left(\frac{1}{2}A\right)$$

(ii) Projection Formula:

Theorem 12.

$a = b\cos C + c\cos B;$

$b = c\cos A + a\cos C;$

$c = a\cos B + b\cos A.$

(iii) Standard Results:

(I) Half-angle formulae:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\Delta};$$

$$\cot\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{s(s-a)}{\Delta}.$$

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C;$$

$$\Delta = \frac{abc}{4R} = rs.$$

(II) Values of sin A, cos A, cot A:

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc};$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc};$$

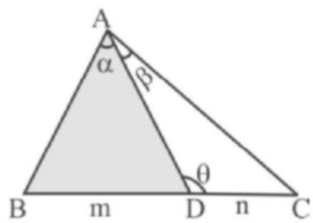
$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{4\Delta}.$$

(III) m-n Theorem:

Theorem:

(1) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(2) $(m+n)\cot\theta = n\cot B - m\cot C$



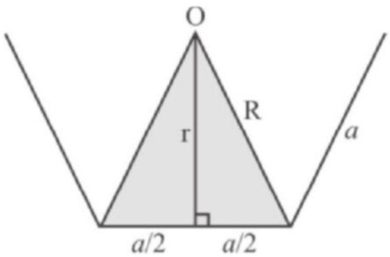
(IV) Relation between inradius, sides, semi-perimeter and area of the triangle:

In radius	r	$\frac{\Delta}{s}$	$(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$	$\frac{a\sin B/2.\sin C/2}{\cos A/2}$
Ex radius (opposite to A)	r_1	$r_1=\frac{\Delta}{s-a}$	$s\tan\frac{A}{2}$	$\frac{a\cos B/2.\cos C/2}{\cos A/2}$
Ex radius (opposite to B)	r_2	$r_2=\frac{\Delta}{s-b}$	$s\tan\frac{B}{2}$	$\frac{b\cos A/2.\cos C/2}{\cos B/2}$
Ex radius (opposite to C)	r_3	$r_3=\frac{\Delta}{s-c}$	$s\tan\frac{C}{2}$	$\frac{c\cos A/2.\cos B/2}{\cos C/2}$

(V) Regular n sides Polygon:

If the polygon has 'n' sides, Sum of the internal angles is $(n-2)\pi$ and each angle is $\frac{(n-2)\pi}{n}$.

a = side length; r = in radius; R = circum-radius



$$r=\frac{a}{2\tan\frac{\pi}{n}} \quad and \quad R=\frac{a}{2\sin\frac{\pi}{n}}$$

Area of polygon $=\frac{1}{4}na^2.\cot\left(\frac{\pi}{n}\right)=nr^2\tan\left(\frac{\pi}{n}\right)=\frac{n}{2}R^2\sin\frac{2\pi}{n}$.

Heights and Distance

Horizontal Ray: A ray parallel to the surface of the earth emerging from the eye of an observer is called a horizontal ray.

Ray of Vision: The ray from the eye of an observer towards the object is called the ray of vision or ray of sight.

Angle of Elevation: If the object under observation is above an observer, but not directly above the observer, then the angle formed by the horizontal ray and the ray of sight in a vertical plane is called the angle of elevation.

Angle of Depression: If the object under observation is at a lower level than an observer but not directly under the observer, then the angle formed by the horizontal ray and the ray of sight is called the angle of depression.

Exercise

Sets & Relations

1. Let $f: R \to R$ be defined as $f(x) = x^4$. Choose the correct option
(a) f is neither one-one nor onto
(b) f is one-one-but not onto
(c) f is many one onto
(d) f is one-one onto

2. In an election, two persons A and B contested. $x\%$ of the total voter voted for A and $(x+20)\%$ for B. If 20 % of the voters did not vote, then $x =$
(a) 30 (b) 25 (c) 40 (d) 35

3. If sets A and B are defined as

$A = \{(x, y) : y = e^x, x \in R\}$

$B = \{(x, y) : y = x, x \in R\}$, then

(a) $B \subset A$ (b) $A \subset B$
(c) $A \cap B = \phi$ (d) $A \cup B = A$

4. If the function $f: R \to R$ is defined by

$f(x) = (x^2 + 1)^{35} \ \forall x \in R$, then f is

(a) Onto but not one-one
(b) Both one - one and onto
(c) Neither one-one nor onto
(d) One-one but not onto

5. Let $A = \{1, 2, 3\}$. The total number of distinct relations that can be defined over A is
(a) 2^9 (b) 6
(c) 8 (d) None of these

6. f and g are two functions such that $(fg)(x) = (gf)(x)$ for all x. Then f and g may be defined as

(a) $f(x) = x^3, g(x) = x + 1$

(b) $f(x) = x^m, g(x) = x^n$ where m, n are unequal integers

(c) $f(x) = \sqrt{x}, g(x) = \cos x$

(d) $f(x) = x - 1, g(x) = x^2 + 1$

Relations & Functions

7. Let a relation R in the set N of natural numbers be defined as

$(x, y) \Leftrightarrow x^2 - 4xy + 3y^2 = 0, \forall x, y \in N$.

The relation R is
(a) Reflexive (b) Symmetric
(c) Transitive (d) An equivalence relation

8. The relation R defined in the set

$A = \{1, 2, 3, 4, 5, 6, 7\}$ by

$R = \{(a, b) : \text{both a and b are either odd or even}\}$.

Then R is
(a) Symmetric (b) Transitive
(c) An equivalence relation (d) Reflexive

9. A function $f: R \to R$ is defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in Q \\ -1 & \text{if} \quad x \in (R - Q) \end{cases}$$

The value of $f(\pi) - f\left(\frac{22}{7}\right)$ is

(a) 2 (b) 0
(c) –2 (d) None of these

10. Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a function from Z to Z defined by $f(x) = ax + b$ for some integers a and b. Then, the values of a and b are

(a) $a = 2, b = -1$ (b) $a = -2, b = 1$
(c) $a = -1, b = 2$ (d) $a = -1, b = -1$

11. The domain of the function

$f(x) = \dfrac{1}{\sqrt{\{\sin x\} + \{\sin x(\pi + x)\}}}$, where $\{.\}$ denotes the fractional part, is

(a) $(2n+1)\pi/2, n \in Z$ (b) $(0, \pi)$
(c) $[0, \pi]$ (d) None of these

12. If $f(x) = \dfrac{1-x}{1+x}$, then $f[f(\cos x)] =$

(a) $\dfrac{1 - \cos x}{1 + \cos x}$ (b) x

(c) $\cos x$ (d) $\tan^2 \dfrac{x}{2}$

Trigonometry

13. A right triangle has perimeter of length 7 and hypotenuse of length 3. If θ is the largest non-right angle in the triangle, then the value of $\cos\theta$ equals

(a) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (b) $\frac{4+\sqrt{2}}{6}$

(c) $\frac{4-\sqrt{2}}{3}$ (d) $\frac{4-\sqrt{2}}{6}$

14. Which one of the following is not correct ?

(a) $|\sin x| \le 1$

(b) $-1 \le \cos x \le 1$

(c) $|\sec x| < 1$

(d) $\cos ec\, x \ge 1$ or $\cos ec\, x \le -1$

15. If $\tan\theta + \sec\theta = e^x$, then $\cos\theta$ equals

(a) $\frac{(e^x+e^{-x})}{2}$ (b) $\frac{2}{(e^x+e^{-x})}$

(c) $\frac{(e^x-e^{-x})}{2}$ (d) $\frac{(e^x-e^{-x})}{(e^x+e^{-x})}$

16. If $0<\phi<\frac{\pi}{2}, x=\sum_{n=0}^{\infty}\cos^{2n}\phi, y=\sum_{n=0}^{\infty}\sin^{2n}\phi$ and

$z=\sum_{n=0}^{\infty}\cos^{2n}\phi\cdot\sin^{2n}\phi$ then

(a) $xyz = xz + y$ (b) $xyz = xy + z$

(c) $xyz = x + y + z$ (d) $xyz = yz + x$

17. If $|\tan x| \le 1$ and $x \in [-\pi, \pi]$ then the solution set for x is

(a) $\left[-\pi,-\frac{3\pi}{4}\right]\cup\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4},\pi\right]$

(b) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4},\pi\right]$

(c) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

(d) None of these

18. If $\max_{\theta\in R}\{5\sin\theta+3\sin(\theta-\alpha)\}=7$ then the set of possible values of α is

(a) $\left\{x \middle| x=2n\pi\pm\frac{\pi}{3}; n\in Z\right\}$

(b) $\left\{x \middle| x=2n\pi\pm\frac{2\pi}{3}; n\in Z\right\}$

(c) $\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$

(d) None of these

Inverse Trigonometry

19. The value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{12}$ (c) $\frac{3\pi}{4}$ (d) $\frac{13\pi}{12}$

20. The domain of $\log_e \sin^{-1}(x)$ is

(a) $(0,1]$ (b) $(0,2]$

(c) $(0,\infty)$ (d) $(-\infty,0]$

21. If $\cos^{-1}x > \sin^{-1}x$, then

(a) $-1 \le x < \frac{1}{\sqrt{2}}$ (b) $x > 0$

(c) $\frac{1}{\sqrt{2}} < x < 1$ (d) $0 \le x < \frac{1}{\sqrt{2}}$

22. Total number of ordered pairs (x, y) satisfying $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$ where $|x| \le 3\pi$, is equal to

(a) 2 (b) 4 (c) 6 (d) 8

23. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)$ is equal to

(a) $\frac{3\pi}{4}$ (b) $\sqrt{2}-1$

(c) $\frac{\pi}{4}$ (d) None of these

24. Number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$ where $-\pi \le x \le \pi$

(a) 0 (b) 1 (c) 2 (d) 4

25. In the following equation, where a and b are coprime positive integers, what is the sum of a and b ?

$$\cos^{-1}\left(\frac{17}{\sqrt{1130}}\right)=\tan^{-1}\left(\frac{a}{b}\right)$$

(a) 29 (b) 17
(c) 46 (d) None of these

Applications of Trigonometry

26. In ΔABC, if $\sin A, \sin B$ are the roots of $c^2x^2-c(a+b)x+ab=0$ then $\sin C=$

(a) 0 (b) 1/2 (c) $1/\sqrt{2}$ (d) 1

27. In ΔABC, the value of

$$(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2} \text{ is}$$

(a) a^2 (b) b^2 (c) c^2 (d) a^2-b^2

28. A circle is circumscribed on an equilateral Triangle ABC where AB = 6 cm. The area of the circumcircle is $K\pi cm^2$. Find the value of K.

(a) 1 (b) 5 (c) 12 (d) 10

29. In ΔABC, if $c^2=a^2+b^2, 2s=a+b+c$, then $4s(s-a)(s-b)(s-c)$

(a) s^4 (b) b^2c^2 (c) c^2a^2 (d) a^2b^2

30. $\sin\theta+\cos\theta=\sqrt{2}\sin(90-\theta), \cot\theta=?$ your answer the nearest integer

(a) 2 (b) 3 (c) 4 (d) 1

31. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is

(a) $\sqrt{\frac{x^3}{8}}$ (b) $\frac{1}{2}x^2$ (c) πx^2 (d) $\frac{3}{2}x^2$

32. In ΔABC, if $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are in H.P. Then, a, b and c will be in

(a) AP (b) GP
(c) HP (d) None of these

ANSWER KEY

1. a	2. a	3. c	4. c	5. a
6. b	7. a	8. c	9. c	10. a
11. d	12. c	13. d	14. c	15. d
16. d	17. a	18. a	19. c	20. a
21. a	22. c	23. a	24. c	25. c
26. d	27. c	28. c	29. d	30. a
31. b	32. c			

CLASS XII MATHEMATICS KVPY-6

PREVIOUS YEAR QUESTIONS

INTEGRAL CALCULUS

1. Let $f(x)=\max\left\{3,x^2,\frac{1}{x^2}\right\}$ for $\frac{1}{2}\le x\le 2$ .Then the value of the integral $\int_{1/2}^{2} f(x)\,dx$ is **[2017]**

(a) $\frac{11}{3}$ (b) $\frac{13}{3}$ (c) $\frac{14}{3}$ (d) $\frac{16}{3}$

2. The number of continuous functions $f:[0,1]\to R$ that satisfy $\int_0^1 x\,f(x)\,dx=\frac{1}{3}+\frac{1}{4}\int_0^1 (f(x))^2\,dx$ is **[2017]**

(a) 0 (b) 1
(c) 2 (d) infinity

3. The area of the region bounded by the curve $y=\left|x^3-4x^2+3x\right|$ and the $x-$axis, $0\le x\le 3$, is **[2016]**

(a) $\frac{37}{6}$ (b) $\frac{9}{4}$ (c) $\frac{37}{12}$ (d) 0

4. The number of continuous function $f:[0,1]\to[0,1]$ such that $f(x)< x^2$ for all x and $\int_0^1 f(x)\,dx=\frac{1}{3}$ is: **[2016]**

(a) 0 (b) 1 (c) 2 (d) infinite

5. On the real line R, we define two functions f and g as follows :

$f(x)=\min\{x-[x],1-x+[x]\}$;

$g(x)=\max\{x-[x],1-x+[x]\}$.

where $[x]$ denotes the largest integer not exceeding x. The positive integer n for which $\int_0^n (g(x)-f(x))\,dx=100$ is **[2016]**

(a) 100 (b) 198 (c) 200 (d) 202

6. Define a function $f:R\to R$ by $f(x)=\max\{|x|,|x-1|,\cdots|x-2n|\}$ where n is a fixed natural number. Then $\int_0^{2n} f(x)\,dx$ is **[2015]**

(a) n (b) n^2 (c) $3n$ (d) $3n^2$

7. If $p(x)$ is a cubic polynomial with $p(1)=3,\ p(0)=2$ and $p(-1)=4$,then $\int_{-1}^{1} p(x)\,dx$ is **[2015]**

(a) 2 (b) 3 (c) 4 (d) 5

8. Let $x>0$ be a fixied real number. Then the integral $\int_0^\infty e^{-t}\left|x-t\right|$ dt is equal to **[2015]**

(a) $x+2e^{-x}-1$ (b) $x-2e^{-x}+1$
(c) $x+2e^{-x}+1$ (d) $-x-2e^{-x}+1$

9. Let $f:R\to R$ be a continuous function satisfying $f(x)+\int_0^x tf(t)\,dt+x^2=0$ .for all $x\in R$ .Then **[2015]**

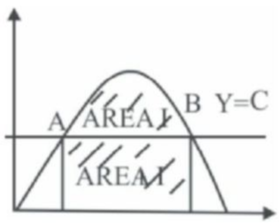
(a) $\lim_{x\to\infty} f(x) = 2$

(b) $\lim_{x\to-\infty} f(x) = -2$

(c) $f(x)$ has more than one point in common with the $x-$axis

(d) $f(x)$ is an odd functions

10. The figure shows a portion of the graph $y = 2x - 4x^3$. The line $y = c$ is such that the areas of the regions marked I and II are equal .If a,b are the $x-$coordinates of A,B respectively ,then $a+b$ equals **[2015]**



(a) $\frac{2}{\sqrt{7}}$ (b) $\frac{3}{\sqrt{7}}$ (c) $\frac{4}{\sqrt{7}}$ (d) $\frac{5}{\sqrt{7}}$

11. Let $f:[0,1]\to[0,\infty)$ be a continuous function such that $\int_0^1 f(x)\,dx = 10$. Which of the following statements is NOT necessarily true? **[2014]**

(a) $\int_0^1 e^{-x} f(x)\,dx \le 10$

(b) $\int_0^1 \frac{f(x)}{(1+x)^2} dx \le 10$

(c) $-10 \le \int_0^1 \sin(100x) f(x)\,dx \le 10$

(d) $\int_0^1 f(x)^2\,dx \le 100$

12. A continuous function $f: R\to R$ satisfies the equation $f(x) = x + \int_0^x f(t)\,dt$ .Which of the following options is true? **[2014]**

(a) $f(x+y) = f(x) + f(y)$

(b) $f(x+y) = f(x)f(y)$

(c) $f(x+y) = f(x) + f(y) + f(x)f(y)$

(d) $f(x+y) = f(xy)$

13. For a real number x let $[x]$ denote the largest integer less than or equal to x and $\{x\} = x - [x]$ .Let n be a positive integer .Then $\int_0^n \cos\left(2\pi[x]\{x\}\right) dx$ is equal to **[2014]**

(a) 0 (b) 1 (c) n (d) $2n-1$

14. Let $I_n = \int_0^{\pi/2} x^n \cos x\,dx$,where n is non-negative integer. Then $\sum_{n=2}^{\infty}\left(\frac{I_n}{n!} + \frac{I_{n-2}}{(n-2)!}\right)$ equals **[2014]**

(a) $e^{\frac{\pi}{2}} - 1 - \frac{\pi}{2}$ (b) $e^{\frac{\pi}{2}} - 1$

(c) $e^{\frac{\pi}{2}} - \frac{\pi}{2}$ (d) $e^{\frac{\pi}{2}}$

15. For a real number x let $[x]$ denote the largest integer less than or equal to x .The smallest positive integer n for which the integral $\int_1^n [x]\left[\sqrt{x}\right] dx$ exceeds 60 is **[2014]**

(a) 8 (b) 9 (c) 10 (d) $[60^{2/3}]$

16. For $x,t \in R$ let $p_t(x) = (\sin t)x^2 - (2\cos t)x + \sin t$ be a family of quadratic polynomials in x with variable coefficients. Let $A(t) = \int_0^1 p_t(x)dx$. Which of the following statements are true? **[2013]**

I. $A(t) < 0$ for all t

II. $A(t)$ has infinitely many critical points

III. $A(t) = 0$ for infinitely many t

IV. $A(t) < 0$ for all t

(a) I and II only (b) II and III only

(c) III and IV only (d)IV and I only

17. For real x with $-10 \le x \le 10$ define $f(x) = \int_{-10}^{x} 2^{[t]} dt$, where for a real number r we

denote by $[r]$ the largest integer less than or equal to r. The number of points of discontinutiy of in the interval $(-10,10)$ is **[2013]**

(a) 0 (b) 10 (c) 18 (d) 19

18. For a real number x let $[x]$ denote the largest integer less than or equal to x and $\{x\} = x - [x]$.
Then smallest possible integer value of n for which $\int_1^n [x]\{x\}\,dx$ exceeds 2013 is **[2013]**

(a) 63 (b) 64 (c) 90 (d) 91

19. Let n be a positive integer. For a real number x, let $[x]$ denote the largest integer not exceeding x and $\{x\} = x - [x]$. Then $\int_1^{n+1} \frac{(\{x\})^{[x]}}{[x]}dx$ is equal to **[2013]**

(a) $\log_e(n)$ (b) $\frac{1}{n+1}$

(c) $\frac{n}{n+1}$ (d) $1+\frac{1}{2}+\cdots+\frac{1}{n}$

20. The value $\dfrac{\int_0^{\pi/2}(\sin x)^{\sqrt{2}+1}dx}{\int_0^{\pi/2}(\sin x)^{\sqrt{2}-1}dx}$ is **[2012]**

(a) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ (b) $\frac{\sqrt{2}-1}{\sqrt{2}+1}$

(c) $\frac{\sqrt{2}+1}{\sqrt{2}}$ (d) $2-\sqrt{2}$

21. The value of $\int_{-2012}^{2012}\left(\sin\left(x^3\right)+x^5+1\right)dx$ is **[2012]**

(a) 2012 (b) 2013 (c) 0 (d) 4024

22. Let $[x]$ and $\{x\}$ be the integer part and fractional part of a real number x respectively. The value of the integral $\int_0^5 [x]\{x\}\,dx$ is **[2012]**

(a) $\frac{5}{2}$ (b) 5 (c) 34.5 (d) 35.5

23. The value of the integral $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$, where $a > 0$ is **[2012]**

(a) π (b) $a\pi$ (c) $\frac{\pi}{2}$ (d) 2π

24. Consider

$L = \sqrt[3]{2012} + \sqrt[3]{2013} + \cdots + \sqrt[3]{3011}$

$R = \sqrt[3]{2013} + \sqrt[3]{2014} + \cdots + \sqrt[3]{3012}$

and $I = \int_{2012}^{3012}\sqrt[3]{x}\,dx$ Then **[2012]**

(a) $L+R<2I$ (b) $L+R>2I$

(c) $L+R=2I$ (d) $\sqrt{LR}=I$

25. Let $f:(2,\infty)\to N$ be defined by $f(x)=$ the largest prime factor of $[x]$. Then $\int_2^8 f(x)\,dx$ is equal to **[2011]**

(a) 17 (b) 22 (c) 23 (d) 25

26. Let $[x]$ denote the largest integer not exceeding x and $\{x\} = x - [x]$.

Then $\int_0^{2012}\frac{e^{\cos(\pi\{x\})}}{e^{\cos(\pi\{x\})}+e^{-\cos(\pi\{x\})}}dx$ is equal to **[2011]**

(a) 0 (b) 1006
(c) 2012 (d) 2012π

27. The value of

$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right)$ is **[2011]**

(a) $\frac{1}{4}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

28. Let $f(x)=x^3+ax^2+bx+c$, where a,b,c are real numbers. If $f(x)$ has a local minimum at $x=1$ and a local maximum at $x=-\frac{1}{3}$ and $f(2)=0$, then $\int_{-1}^{1} f(x)\,dx$ equals **[2011]**

(a) $\frac{14}{3}$ (b) $\frac{-14}{3}$ (c) $\frac{7}{3}$ (d) $\frac{-7}{3}$

29. For each positive integer *n*, define

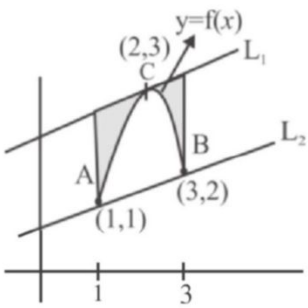
$f_n(x) = \text{minimum}\left(\frac{x^n}{n!}, \frac{(1-x)^n}{n!}\right)$, for $0 \le x \le 1$. Let

$I_n = \int_0^1 f_n(x)\,dx, n \ge 1$ Then $I_n = \sum_{n=1}^{\infty} I_n$ is equal to **[2011]**

(a) $2\sqrt{e}-3$ (b) $2\sqrt{e}-2$

(c) $2\sqrt{e}-1$ (d) $2\sqrt{e}$

30. The following figure shown the graph of a continuous function $y = f(x)$ on the interval [1,3].The points A, B, C have coordinates (1,1), (3,2),(2,3) respectively , and the lines L_1 and L_2 are parallel , with L_1 being tangent to the curve at C. If the area under the graph of $y = f(x)$ from $x=1$ to $x=3$ is 4 square units, then the area of the shaded region is **[2010]**



(a) 2 (b) 3 (c) 4 (d) 5

31. Let $I_n = \int_0^1 (\log x)^n\,dx$,where *n* is a non -negative integer. Then $I_{2001} + 2011\, I_{2010}$ is equal to **[2010]**

(a) $I_{100} + 999\ I_{998}$ (b) $I_{890} + 890\ I_{889}$

(c) $I_{100} + 100\ I_{99}$ (d) $I_{53} + 54\ I_{52}$

32. Consider the region $A = \{(x,y)\ |x_2 + y_2 \ge 100\}$ and $= \{(x,y)\ |\sin(x+y) > 0\}$ in the plane. Then the area of the region $A \cap B$ is **[2010]**

(a) $10\,\pi$ (b) 100 (c) $100\,\pi$ (d) $50\,\pi$

33. Let $f: R \to R$ be a continuous function satisfying $f(x) = x + \int_0^x f(t)dt$, for all $x \in R$ .Then the number of elements in the set $S = \{x \in R; f(x) = 0\}$ is **[2010]**

(a) 1 (b) 2 (c) 3 (d) 4

34. The value of $\int_0^{2\pi} \min\{|x-\pi|, \cos^{-1}(\cos x)\}\,dx$ is **[2010]**

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{8}$ (d) π^2

ANSWER KEY

1. c	2. b	3. c	4. a	5. c
6. d	7. d	8. a	9. b	10. a
11. d	12. c	13. c	14. a	15. b
16. b	17. a	18. d	19. c	20. d
21. d	22. b	23. c	24. a	25. b
26. b	27. d	28. b	29. a	30. a
31. c	32. d	33. a	34. b	

HINTS & SOLUTIONS

1.Sol: Rewrite the given function as

$$f(x) = \begin{cases} \frac{1}{x^2}; & \frac{1}{2} \le x \le \frac{1}{\sqrt{3}} \\ 3\ ; & \frac{1}{3} \le x \le \sqrt{3} \\ x^2\ ; & \sqrt{3} \le x \le 2 \end{cases}$$

Therefore the given integral is written as

$$\int_{\frac{1}{2}}^{2} f(x)\,dx = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{1}{x^2}\,dx + \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} 3\,dx + \int_{\sqrt{3}}^{\sqrt{3}} 3\,dx + \int_{\sqrt{3}}^{2} x^2\,dx$$

$$= -\sqrt{3} + 2 + 2\sqrt{3} + \frac{8}{3} - \sqrt{3} = \frac{14}{3}$$

2.Sol: Given integral is rewritten as

$$\frac{1}{3} = \int_0^1 x\,f(x)\,dx - \frac{1}{4}\int_0^1 (f(x))^2\,dx$$

$$= \int_0^1 \left(x\,f(x) - \frac{1}{4}(f(x))^2 \right) dx$$

$$= \int_0^1 \left[-\left(\frac{f(x)}{2} - x \right)^2 + x^2 \right] dx$$

$$= -\int_0^1 \left(\frac{f(x)}{2} - x \right)^2 dx + \frac{1}{3}$$

Therefore $-\int_0^1 \left(\frac{f(x)}{2} - x \right)^2 dx = 0$

3.Sol: The given modulus function is rewritten as

$$y = \begin{cases} x^3 - 4x^2 + 3x \ ; & 0 \le x < 1 \\ -x^3 + 4x^2 - 3x \ ; & 1 \le x \le 3 \end{cases}$$

The area of above function can be find out using definite integral . That is

$$\int_0^1 \left(x^3 - 4x^2 + 3x \right) + \int_1^3 \left(-x^3 + 4x^2 - 3x \right) dx$$

$$= \left[\frac{x^4}{3} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_1^0 - \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_1^3$$

$$= \frac{37}{12}$$

4.Sol: We have $f:[0,1] \to [0,1]$, which means $f(x) > 0$. Since $f(x) < x^2$, therefore

$\int_0^1 f(x)\,dx < \int_0^1 x^2 dx$, which yields $\int_0^1 f(x)\,dx \le \frac{1}{3}$.

But we have given that $\int_0^1 f(x)\,dx = \frac{1}{3}$. So no function is continuous with the given condition.

5.Sol: From the graphs it is clear that both $f(x)$ and $g(x)$ are periodic functions. So we have

$\int_0^n g(x)\,dx = n\int_0^1 g(x)\,dx$ and $\int_0^n f(x)\,dx = n\int_0^1 f(x)\,dx$

and from the graph, we have

$$\int_0^1 f(x)\,dx = \frac{1}{2}\cdot 1 \cdot \frac{1}{2} = \frac{1}{4} \text{ and}$$

$$\int_0^1 g(x)\,dx = \frac{1}{2}\cdot 1 \cdot \frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3}{4}.$$

Therefore $\int_0^n g(x)\,dx - \int_0^n f(x)\,dx = \frac{3}{4}n - \frac{1}{4}n = 100,$

$$\Rightarrow \qquad \frac{n}{2} = 100.$$

i.e., $\qquad n = 200.$

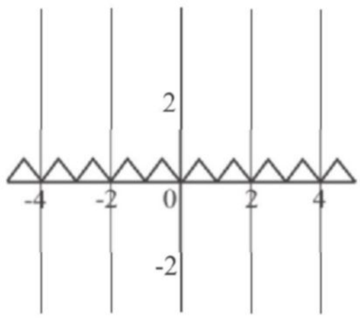


Fig 1. $\text{Min}\left(x - [x], 1 - x + [x] \right)$

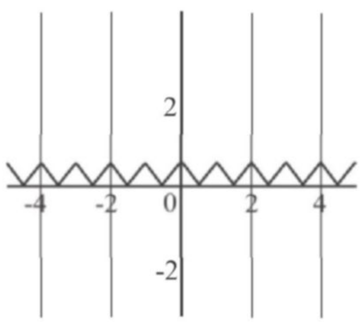


Fig 2. $\text{Min}\left(x - [x], 1 - x + [x] \right)$

6.Sol: Rewrite the given function as

$$f(x) = \begin{cases} |x - 2n| \ ; & x < n \\ |x| \ ; & x \ge n \end{cases}$$

Therefore the given integral is

$$\int_0^{2n} f(x)\,dx = \int_0^n |x-2n|\,dx + \int_n^{2n} |x|\,dx$$

$$= \int_0^n (2n-x)\,dx + \int_n^2 nx\,dx$$

$$= \left[2nx - \frac{x^2}{2}\right]_0^{2n} + \left[\frac{x^2}{2}\right]_n^{2n}$$

$$= \frac{3n^2}{2} + \frac{3n^2}{2} = 3n^2$$

7.Sol: As the integration range is symmetric, we can only care about the even part of the polynomial,

$$e(x) = \frac{p(x)+p(-x)}{2} = bx^2 + d$$

giving the integral

$$\int_{-1}^{1} (bx^2+d)\,dx = 2\int_0^1 (bx^2+d)\,dx.$$

But we have $e(0) = d = 2$ and

$$e(1) = \frac{p(1)+p(-1)}{2} = \frac{3+4}{2},$$

i.e., $\qquad b + d = \frac{7}{2}$

and $\qquad b = \frac{3}{2}.$

Now we have

$$\int_{-1}^{1} (bx^2+d)\,dx = 2\int_0^1 \left(\frac{3}{2}x^2+2\right)dx$$

$$= 2\left[\frac{3}{2}\frac{x^3}{3} + 2x\right]_0^1 = 2\cdot\frac{3}{2} = 3.$$

8.Sol: Given integral is rewritten as

$$f(x) = \int_0^\infty e^{-t}|x-t|\,dt$$

$$= \int_0^x e^{-t}(x-t)\,dt + \int_x^\infty e^{-t}(t-x)\,dt$$

$$f'(x) = \left[x\left(-e^{-x}+e^{-0}\right) + \left(e^{-t}t + e^{-t}\right)_0^x\right] + \int_x^\infty e^{-t}(t-x)\,dt$$

$$= 1 - e^{-x} + \left[e^{-t}\right]_x^\infty = 1 - 2e^{-x}$$

Now $f(x) = x + 2e^{-x} + C,$ and we have

$$f(0) = \int_0^\infty te^{-t}dt = 1.$$

i.e., $\qquad c = -1.$

Hence $\qquad f(x) = x + 2e^{-1} - 1.$

9.Sol: $f(x) + \int_0^x t\,f(t)\,dt + x^2 = 0$

Now differentiating wrt 'x', we get

$$f'(x) + xf(x) + 2x = 0$$

i.e, $\qquad \frac{dy}{dx} + xy = -2x$

$$\frac{dy}{dx}e^{\frac{x^2}{2}} + xy\,e^{\frac{x^2}{2}} = -2x\,e^{\frac{x^2}{2}}$$

up on integration on both sides, we get

$$ye^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}}(-2x)\,dx + c$$

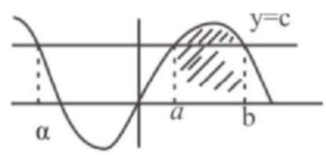
i.e., $\qquad e^{\frac{x^2}{2}}y = -2e^{-\frac{x^2}{2}} + c$

we have $f(0) = 0 \Rightarrow c = 2$

$\therefore \qquad f(x) = 2\left[e^{-\frac{x^2}{2}} - 1\right]$

$\Rightarrow \qquad \lim_{x\to-\infty} f(x) = -2$

10.Sol:



Given that, Area of region marked I and II are equal

i.e., $\qquad \int_a^b (2x - 4x^3)dx = 2(b-a)\,c$

$$\left[x^2 - x^4\right]_a^b = 2(b-a)\,c$$

$\Rightarrow \qquad (a+b)(1-(a^2+b^2)) = 2c$

$(a+b)(1-(a+b)^2 + 2ab) = 2c \qquad (1)$

Also given that $y = c$

i.e., $$2x-4x^3=c$$

$\Rightarrow$ $$4x^3-2x+c=0$$

Now roots of $4x^3-2x+c=0$ are a,b,α

i.e., $$a+b+\alpha=0 \qquad (2)$$

$$ab+(a+b)\alpha=\frac{-1}{2} \qquad (3)$$

$\Rightarrow$ $$7\alpha^3=4\alpha$$

i.e., $$\alpha=\frac{2}{\sqrt{7}}$$

11.Sol: Let $$I=\int_0^1 f(x)^2\,dx.$$

Now put $x^2=t$, and differentiate we get

$dx=\dfrac{dt}{2\sqrt{t}}$ and substitute it in the above integral,

we get $$I=\int_0^1 f(t)\frac{dt}{2\sqrt{t}}.$$

We know $\dfrac{1}{2}\le\dfrac{1}{2\sqrt{t}}$, which is rewritten as

$$\int_0^1 f(t)\frac{dt}{2}\le\int_0^1 f(t)\frac{dt}{2\sqrt{t}}$$

That is $I\ge 5$. But we are given that $I\le 100$. Therefore the option (d) is not true, as there is chance of taking values of $I\ge 100$.

12.Sol: Using Leibniz's rule ,we can write the given integral as

$$f'(x)=1+f(x).$$

i.e., $$\frac{f'(x)}{1+f(x)}=1$$

up on integration we get

$$\ln(1+f(x))=x$$

$\Rightarrow$ $$1+f(x)=e^x$$

So $f(x)=e^x-1$. Now verify all the option with derived $f(x)$, we find that option (c) is correct.

13.Sol: We know $\cos\theta$ is a periodic function. So the given integral is rewritten as

$$\int_0^n \cos(2\pi[x]\{x\})\,dx=n\int_0^1 \cos(2\pi[x]\cdot 1)\,dx$$

$$=n\int_0^1 1\,dx=n$$

Since $[x]=0$ in the interval $[0,1]$, implies that $\{x\}=1$.

14.Sol: we have, $I_n=\int x^n\cos x\,dx$ then

$$I_n=x^n\sin x+nx^{n-1}\cos x-n(n-1)I_{n-2}$$

Now the given integral is rewritten as

$$I_n+n(n-1)I_{n-2}=\left(\frac{\pi}{2}\right)^n$$

Now $$\sum_{n=2}^{\infty}\left(\frac{I_n}{n!}+\frac{I_{n-2}}{(n-2)!}\right)=\sum_{n=2}^{\infty}\left[\frac{I_n+n(n-1)I_{n-2}}{n!}\right]$$

$$=\sum_{n=2}^{\infty}\frac{\left(\frac{\pi}{2}\right)^n}{n!}$$

$$=e^{\frac{\pi}{2}-1}-1-\frac{\pi}{2}$$

15.Sol: Let $I=\int_1^n [x]\left[\sqrt{x}\right]dx$

we define $\left[\sqrt{x}\right]$ in the following manner

$$\begin{aligned} [x]&=1; && 1\le x<4\\ [x]&=2; && 4\le x<9\\ [x]&=3 && 9\le x<16\\ &\vdots && \vdots \end{aligned}$$

Now, the given integral is

$$I=\int_1^2 dx+\int_2^3 2dx+\int_3^4 3dx+\int_4^5 8dx+\int_5^6 10dx+\int_6^7 12dx$$

$$+\int_7^8 14dx+\int_8^9 16dx+\int_9^{10} 27dx+\int_{10}^{11} 30dx\cdots$$

$$I=1+2+3+8+10+12+14+16=66$$

$\therefore \quad n = 9$

16.Sol: $A(t) = \int_0^1 \{(\sin t)x^2 - (2\cos t)x + \sin t\} dx$

$$A(t) = \frac{\sin t}{3} - \cos t + \sin t = \frac{4}{3}\sin t - \cos t$$

Now differentialting A(t) wrt (t) we get

$$A'(t) = \frac{4\cos t}{3} + \sin t$$

17.Sol: Let r be an integer in $(-10, 10)$

To check the continuity of a functions, we need to verify $\lim_{x \to a} f(x) = f(a)$. That is

$$\text{Now LHL} = \lim_{x \to r^-} \int_{-10}^{x} 2^{[t]} dt$$

$$\lim_{h \to 0^-} \left[\int_{-10}^{-9} 2^{[t]} dt + \int_{-9}^{-8} 2^{[t]} dt + \cdots + \int_{r-1}^{r-h} 2^{[t]} dt \right]$$

$$= \lim_{h \to 0} [2^{-10} + 2^{-9} + \cdots + 2^{r-1}(1-h)]$$

$$= 2^{-10} + 2^{-9} + \cdots + 2^{r-1} \qquad (1)$$

$$RHL = \lim_{x \to r^+} \int_{-10}^{x} 2^{[t]}\, dt$$

$$\lim_{h \to 0} \left[\int_{-10^2}^{-9} 2^{[t]} dt + \int_{-9}^{-8} 2^{[t]} dt + \cdots + \int_{r}^{r+h} 2^{[t]} dt \right]$$

$$= 2^{-10} + 2^{-9} + \cdots + 2^{r-1} \qquad (2)$$

$$\text{and } f(r) = \int_{-10}^{r} 2^{[t]} dt = 2^{-10} + 2^{-9} + \cdots + 2^{r-1} \qquad (3)$$

from eq (1), (2) and (3), we have $f(x)$ is continous at all integers.

18.Sol: As the given integral is piece wise function, we can rewrite it as

$$\int_1^n [x] x\, dx = \sum_{r=1}^{n-1} \int_r^{r+1} r(x-r)\, dx$$

$$= \sum_{r=1}^{n-1} r \left[\frac{x^2}{2} - rx \right]_r^{r+1}$$

$$= \sum_{r=1}^{n-1} r \left(\frac{1}{2} \right)$$

$$= \frac{1}{2} \frac{n(n-1)}{2}$$

also given that $\frac{1}{2}\left(\frac{n(n-1)}{2}\right) \geq 2013$, That is

$$\Rightarrow \qquad \frac{1}{2}\frac{n(n-1)}{2} \geq 2013$$

$$\Rightarrow \qquad \frac{1}{2}\frac{n(n-1)}{2} \geq 2013$$

$$\Rightarrow \qquad n(n-1) \geq 2013 \times 4$$

$$\text{i.e.,} \qquad \left(n - \frac{1}{2}\right)^2 \geq \frac{2013 \times 16 + 1}{4}$$

$$n \geq \frac{\sqrt{32209}}{2} + \frac{1}{2}$$

$$n = 91$$

19 .Sol :

$$\int_1^{n+1} \frac{\{x\}^{[x]}}{[x]} dx = \int_1^2 \frac{\{x\}^{[x]}}{[x]} dx + \int_2^3 \frac{\{x\}^{[x]}}{[x]} dx + \cdots$$

$$\int_1^{n+1} \frac{\{x\}^{[x]}}{[x]} dx = \sum_{r=1}^{n} \int_r^{r+1} \frac{\{x\}^{[x]}}{[x]} dx$$

$$= \sum_{r=1}^{n} \int_r^{r+1} \frac{(x-r)^r}{r} dx$$

$$= \sum_{r=1}^{n} \left[\frac{(x-r)^{r+1}}{r(r+1)} \right]_r^{r+1}$$

$$= \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

20.Sol: Let the top integral be *I*, and the bottom one *J*.

Integrate the top one by parts, letting $du = \sin x$ and $v = \sin^{\sqrt{2}} x$. This is the standard way to get a reduction formula for $\int \sin^n x\, dx\ 3mm$.

So $dv = \sqrt{2}\sin^{\sqrt{2}-1} x\, dx$ and we can take $u = -\cos x$.

Then

$$I = \left[-\cos x \sin^{\sqrt{2}} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sqrt{2}\cos^2 x \sin^{\sqrt{2}-1} x\, dx$$

The first part becomes 0. Now rewrite $\cos^2 x$ as $1-\sin^2 x$. Then $I = \sqrt{2}J - \sqrt{2}I$

Now we get

$$I = \frac{\sqrt{2}}{\sqrt{2}+1} J$$

that is $$\frac{I}{J} = \frac{\sqrt{2}}{\sqrt{2}+1} = 2-\sqrt{2}$$

21.Sol: If $f(x)$ is odd function then $\int_{-a}^{a} f(x)\,dx = 0$

$$\int_{-2012}^{2012} (\sin(x^3)+x^5+1)dx = \int_{-2012}^{2012} 1\, dx = 4024$$

22. Sol: Given integral is rewriten as

$$\int_0^5 [x]x\, dx = \int_0^1 0\cdot x\, dx + \int_1^2 1\cdot x\, dx + \int_2^3 2\cdot x\, dx$$

$$+\int_3^4 4\cdot x\, dx + \int_4^5 4\cdot x\, dx$$

$$= 0 + \int_0^1 x\, dx + 2\int_0^1 x\, dx + 3\int_0^1 x\, dx + 4\int_0^1 x\, dx$$

$$= (1+2+3+4)\int_0^1 x\, dx$$

$$= 10\left[\frac{x^2}{2}\right]_0^1$$

$$= 5$$

23.Sol: $$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \qquad (1)$$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx = \int_{-\pi}^{\pi} a^x \frac{\cos^2(x)}{1+a^x} dx \qquad (2)$$

adding (1) and (2), we get

$$2I = \int_{-\pi}^{\pi} \frac{\cos^2(x)}{1+a^x} + dx + \int_{-\pi}^{\pi} a^x \frac{\cos^2 x}{1+a^x} dx$$

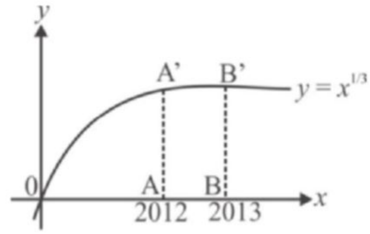
$$= \int_0^{\pi} 2\cos^2 x\, dx$$

That is

$$I = \int_0^{\pi} \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{\pi}{2}$$

24.Sol: Given curve $y = x^{\frac{1}{3}}$, is concave down wards for $x > 0$



Now Area of trepezoid $AA'B'B < \int_{2012}^{2013} x^{\frac{1}{3}} dx$

i.e., $$\frac{(2012)^{\frac{1}{3}} + (2013)^{\frac{1}{3}}}{2}(1) < \int_{2012}^{2013} x^{\frac{1}{3}}\, dx$$

like wise, we can find other strips of trepezoids as

$$\frac{(2013)^{\frac{1}{3}} + (2014)^{\frac{1}{3}}}{2} < \int_{2013}^{2014} x^{\frac{1}{3}}\, dx$$

$$\frac{(2014)^{\frac{1}{3}} + (2015)^{\frac{1}{3}}}{2} < \int_{2014}^{2015} x^{\frac{1}{3}}\, dx$$

$$\vdots \qquad \vdots$$

$$\frac{(3011)^{\frac{1}{3}} + (3012)^{\frac{1}{3}}}{2} < \int_{3011}^{3012} x^{\frac{1}{3}}\, dx$$

Adding all strips of trepezoids, we get

$$\frac{L+R}{2} < \int_{2012}^{3012} x^{\frac{1}{3}}\, dx$$

i.e., $$\frac{L+R}{2} < I$$

$$\Rightarrow \quad L+R < 2I$$

25.Sol: Re write the given integral as

$$\int_2^8 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

$$+\int_5^6 f(x)\,dx + \int_6^7 f(x)\,dx + \int_7^8 f(x)\,dx$$

We know $[x] = 2,3,4,5,6,7$ in the respective

intervals $[2,3],[3,4],[4,5],...,[7,8]$. Therefore the given integral becomes

$$\int_2^8 f(x)\,dx = \int_2^3 2\,dx + \int_3^4 3\,dx + \int_4^5 2\,dx$$

$$+\int_5^6 5\,dx + \int_6^7 3\,dx + \int_7^8 7\,dx$$

$$= 2+3+2+5+3+7 = 22$$

26.Sol: Given function is periodic, so the given integral is rewritten as

$$I = 2012\int_0^1 \frac{e^{\cos \pi x}}{e^{\cos \pi x}+e^{\cos \pi x}}\,dx \qquad (1)$$

$$\therefore \quad I = 2012\int_0^1 \frac{e^{-\cos \pi x}}{e^{-\cos \pi x}+e^{\cos \pi x}}\,dx \qquad (2)$$

adding (1), (2), we get

$$2I = 2012 \Rightarrow I = 1006$$

27.Sol:

$$\lim_{h\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} = \lim_{h\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{4-\left(\frac{r}{n}\right)^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^1 = \frac{\pi}{6}$$

28.Sol: Given that $f(x) = x^3+ax^2+bx+c$, and also $f(x)$ has minimum at $x=1$ and maximum at $x=-\frac{1}{3}$. That is $f'(1)=0=f'\left(-\frac{1}{3}\right)$.

Now, we have $f'(x)=3x^2+2ax+b$, which yields that $a=b=-1$ and also given that $f(2)=0$.

That is $\quad 8-4-2+c=0$

Therefore $c=-2$ and $f(x)=x^3-x^2-x-2$

Given integral is

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x^3-x^2-x-2)\,dx = \frac{-14}{3}$$

29.Sol: $I_n = \int_0^{1/2}\frac{x^n}{n!}\,dx + \int_{1/2}^{1}\frac{(1-x)^n}{n!}\,dx$

$$= \frac{1}{(n+1)!}\left(\left(\frac{1}{2}\right)^{n+1}+\left(\frac{1}{2}\right)^{n+1}\right) = \frac{\left(\frac{1}{2}\right)^n}{(n+1)!}$$

$$\sum_{n=1}^{\infty} I_n = \left(\frac{1/2}{2!}+\frac{(1/2)^2}{3!}+\cdots\right) = 2\sqrt{e}-3$$

30.Sol: Equation of Line L_2 is given as

$$y-1 = \frac{2-1}{3-1}(x-1)$$

i.e., $\quad 2y-x=1$

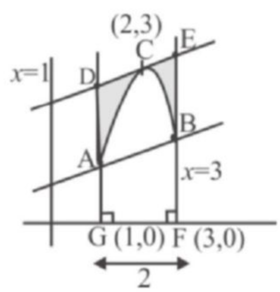
Hence slope of line L_2 is $\frac{1}{2}$

Simillarly, equation of line L_1 is given as

$$y-3 = \frac{1}{2}(x-2)$$

i.e., $\quad 2y-x=4$

Now Co-or dinate of D is $(1,\frac{5}{2})$ and $E(3,\frac{7}{2})$



Area under $f(x)=4$

Now shaded area $=$ Area of trapezium DEFG

-Area under $f(x)$

$$=\frac{1}{2}\left(\frac{5}{2}+\frac{7}{2}\right)\times 2-4 = 6-4 = 2$$

31.Sol: Given integral is

$$I_n = \int_1^e (\log x)^n\,dx$$

$$= x(\log x)^n\Big|_1^e - \int_1^e \frac{n(\log x)^{n-1}}{x}x\,dx$$

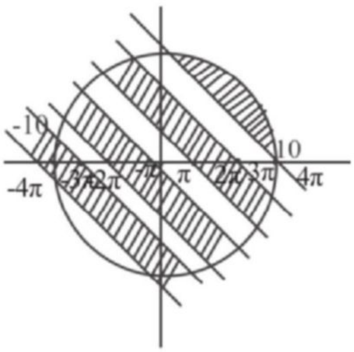
$$= (e-0) - nI_n - 1$$

$$\Rightarrow \qquad I_n + nI_n - 1 = e$$

$$\therefore \qquad I_{2011} + 2011\, I_{2010} = e$$

similarly $\quad I_{100} + 100\, I_{99} = e$

32.Sol: Given that $x^2 + y^2 \le 100$ and $\sin(x+y) > 0$



required area $=$ shaded region

$$= \frac{1}{2}\pi(10)^2 = 50\pi$$

33.Sol: We have

$$f'(x) = 1 + f(x)$$

i.e., $\dfrac{dy}{dx} = 1 + y$

$$\Rightarrow \quad \ln(1+y) = x + c$$

i.e., $\quad y = \lambda e^x - 1$

$$\Rightarrow \qquad f(x) = \lambda e^x - 1 \qquad (1)$$

Now $\quad f(x) = x + \int_0^x f(t)\,dt$

$$\lambda e^x - 1 = x + \lambda\left[e^t - t\right]_0^x$$

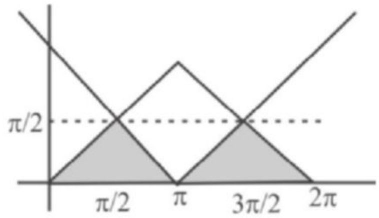
$$\lambda e^x - 1 = x + \lambda e^x - x - \lambda$$

$$\Rightarrow \qquad \lambda = 1$$

$$\therefore \qquad f(x) = e^x - 1$$

for $\quad f(x) = 0$, yields $x = 0$.

34 .Sol:



From the graph, we have

$$I = \int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx + \int_{\pi}^{3\pi/2} (x-\pi)\,dx + \int_{3\pi/2}^{2\pi} (2\pi - x)\,dx$$

$$\frac{\pi^2}{8} + \frac{\pi^2}{8} + \frac{\pi^2}{8} + \frac{\pi^2}{8} = \frac{\pi^2}{2}$$

MOCK TEST PAPER

JEE MAIN - 6

2018 2019

1. Let R be a reflexive relation on a finite set A having n elements, and let there be m ordered pairs in R, then
(a) $m \geq n$ (b) $m \leq n$
(c) $m = n$ (d) $m < n$

2. R is a relation greater than or equal "from A = $\{1,2,3,4\}$ to B = $\{4, 5, 6\}$, then R^{-1} =
(a) $\{(4, 4)\}$ (b) ϕ (c) $A \times B$ (d) R

3. If $f(x) = \begin{cases} x-1, & x<0 \\ \frac{1}{4}, & x=0 \\ x^2, & x>0 \end{cases}$, then
(a) $\lim_{x \to 0^-} f(x) = 1$
(b) $\lim_{x \to 0^+} f(x) = 1$
(c) $f(x)$ is discontinuous at $x = 0$
(d) None of these

4. If the function $f(x) = \frac{k \sin x + 2\cos x}{\sin x + \cos x}$ is strictly increasing for all values of x, then
(a) $k < 1$ (b) $k > 1$ (c) $k < 2$ (d) $k > 2$

5. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is
(a) $y = 0$ (b) $y = 3$
(c) $y = 1$ (d) $y = 2$

6. $\int \frac{dx}{\sin x - \cos x + \sqrt{2}}$ equals :
(a) $-\frac{1}{\sqrt{2}} \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$ (b) $\frac{1}{\sqrt{2}} \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$
(c) $\frac{1}{\sqrt{2}} \cot\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$ (d) $-\frac{1}{\sqrt{2}} \cot\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$

7. $I_n = \int_0^{\pi/4} \tan^n x dx, n \in N$, then $I_{10} + I_8 =$
(a) $\frac{1}{9}$ (b) $\frac{1}{8}$ (c) $\frac{1}{7}$ (d) 9

8. If the vertcies of a triangle have rational coordinates, then the coordinates of which of the following are necessarily rational ?
(a) Centroid (b) Circumcenter
(c) Orthocenter (d) Incenter
(a) A only (b) A and B only
(c) A, B and C only (d) A, B, C and D

9. The points $A(-4, -1)$, $B(-2,-4)$, $C(4,0)$ and $D(2,3)$ are the vertices of
(a) Parallelogram (b) Rectangle
(c) Rhombus (d) None of these

10. The number of integer values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is
(a) 2 (b) 0 (c) 4 (d) 1

11. The number of common tangents to the circles $x^2 + y^2 - y = 0$ and $x^2 + y^2 + y = 0$ is
(a) 2 (b) 3 (c) 0 (d) 1

12. The point on parabola $2y = x^2$, which is nearest to the point (0, 3) is

(a) $(\pm 4, 8)$ (b) $(\pm 1, 1/2)$

(c) $(\pm 2, 2)$ (d) None of these

13. The eccentricity of the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of coordinates is

(a) $\frac{2\sqrt{6}}{7}$ (b) $\frac{3\sqrt{2}}{7}$

(c) $\frac{\sqrt{6}}{7}$ (d) None of these

14. $P(x)$ is a polynomial with integral coefficients such that for four distinct integers a, b, c, d, $P(a) = P(b) = P(c) = P(d) = 3$. If $P(e) = 5$ (e is an integer), then

(a) $e = 1$ (b) $e = 3$

(c) $e = 4$ (d) No real value of e

15. If three distinct real numbers a, b, c satisfy $a^2(a+p) = b^2(b+p) = c^2(c+p)$ where then value of $ab+bc+ca$ is

(a) $-p$ (b) p (c) 0 (d) $\frac{p^2}{2}$

16. If $(1+x)^n = P_0 + P_1x + P_2x^2 + \dots + P_n x^n$, then find the value of the following series:

(i) $P_0 - P_2 + P_4 + \dots$ is:

(a) $2^{n/2}\cos\frac{n\pi}{4}$ (b) $2^{n/2}\sin\frac{n\pi}{4}$

(c) $-2^{n/2}\sin\frac{n\pi}{4}$ (d) $-2^{n/2}\cos\frac{n\pi}{4}$

17. If $n \geq 2$ then

$3\cdot C_0 - 5\cdot C_1 + 7\cdot C_2 - \dots + (-1)^n(2n+3)C_n =$

(a) 0 (b) 3 (c) -3 (d) 1

18. An elevator starts with m passengers and stops at n floors $(m \leq n)$. The probability that no two passengers alight at the same floor is

(a) $\frac{{}^nP_m}{m^n}$ (b) $\frac{{}^nP_m}{n^m}$ (c) $\frac{{}^nC_m}{m^n}$ (d) $\frac{{}^nC_m}{n^m}$

19. A library has a copies of one book, b copies each of two books, c copies each of three books, a single copy of d books. The total number of ways in which these books can be arranged in a shelf is equal to

(a) $\frac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$ (b) $\frac{(a+2b+3c+d)!}{a!(2b!)(c!)^3}$

(c) $\frac{(a+b+3c+d)!}{(c!)^3}$ (d) $\frac{(a+2b+3c+d)!}{a!(2b)!(3c!)}$

20. The triangle ABC is defined by the vertices $A(1,-2,2)$ $B(1,4,0)$ and $C(-4,1,1)$. Let M be the foot of the altitude drawn from the vertex B to side AC. Then $BM =$

(a) (-20/7, -30/7, 10/7) (b) (-20, -30, 10)

(c) (2, 3, -1) (d) None of these

21. For the plane $\pi = 4x - 3y + 2z - 3 = 0$, the points $A = (-2,1,2), B = (3,1,-2)$

(a) Lie on the same side of $\pi = 0$

(b) Lie on the opposite of $\pi = 0$

(c) Lie on the normal to $\pi = 0$

(d) None of these

22. The equation of the sphere concentric with the sphere $x^2 + y^2 + z^2 - 4x - 2y - 6z - 7 = 0$ and passing through (0, 0, 0) is

(a) $x^2 + y^2 + z^2 - 4x - 2y - 6z = 0$

(b) $x^2 + y^2 + z^2 = 0$

(c) $x^2 + y^2 + z^2 - 4x - 2y = 0$

(d) None of these

23. The equation $ax^2 + bx + c = 0$, where a, b, c are the sides of a ΔABC, and the equation $x^2 + \sqrt{2}x + 1 = 0$ have a common root. The measure of $\angle C$ is

(a) 90° (b) 45°

(c) 60° (d) None of these

24. If A is in the III quadrant, $3\tan A - 4 = 0$ then $5\sin 2A + 3\sin A + 4\cos A =$

(a) 0 (b) -24/5 (c) 24/5 (d) 5/24

25. The number of all the possible matrices of order 2×2 with each entry 0,1 or 2 is

(a) 64 (b) 81

(c) 12 (d) None of these

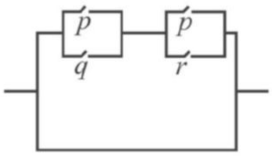
26. If $f(x) = \begin{vmatrix} x-3 & 2x^2-18 & 3x^3-81 \\ x-5 & 2x^2-50 & 4x^3-500 \\ 1 & 2 & 3 \end{vmatrix}$, then

$f(1)\cdot f(3) + f(3)\cdot f(5) + f(5)\cdot f(1)$ is equal to

(a) $f(1)$ (b) $f(3)$

(c) $f(1)+f(3)$ (d) $f(1)+f(3)+f(5)$

27. Simplify the following circuit and find the boolean polynomial.



(a) $p\vee(q\wedge r)$ (b) $p\wedge(q\vee r)$

(c) $p\vee(q\vee r)$ (d) $p\wedge(q\wedge r)$

28. The mean of a set of numbers is $\bar{x}$. If each number is decreased by λ, the mean of the new set is-

(a) $\bar{x}$ (b) $\bar{x}+\lambda$ (c) $\lambda-\bar{x}$ (d) $\bar{x}-\lambda$

29. The sum of 20 terms of the series whose r^{th} term is given by $T(n) = (-1)^n \dfrac{n^2+n+1}{n!}$ is

(a) $\dfrac{20}{19!}-2$ (b) $\dfrac{21}{20!}-1$

(c) $\dfrac{21}{20!}$ (d) None of these

30. Let 'f' be a strict decreasing function with range [a, b]. The domain of the function 'f' is

(a) $\left[f^{-1}(a), f^{-1}(b)\right]$ (b) $[b,a]$

(c) $\left[f^{-1}(b), f^{-1}(a)\right]$ (d) None of these

ANSWER KEY

1. a	2. a	3. c	4. d	5. b
6. d	7. a	8. c	9. b	10. a
11. b	12. c	13. a	14. d	15. c
16. b	17. a	18. b	19. a	20. a
21. b	22. a	23. b	24. a	25. b
26. b	27. a	28. d	29. b	30. c

HINTS & SOLUTIONS

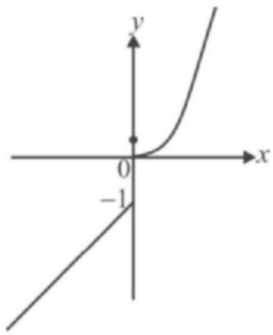
1.Sol: Since R is reflexive relation on A, therefore $(a,a)\in A, \forall a \in A$.

The minimum number of ordered pair in R is n.

Hence, $m \geq n$.

2.Sol: $R = \{(4,4)\}$ and $R^{-1} = \{(4,4)\}$

3.Sol: Given $f(x) = \begin{cases} x-1, & x<0 \\ \dfrac{1}{4}, & x=0 \\ x^2, & x>0 \end{cases}$



Clearly from the graph, we can see $f(x)$ is discontinuous at $x=0$

Aliter

$$f(0^+) = \lim_{h\to 0}(0+h)^2 = 0$$

$$f(0^-) = \lim_{h\to 0}(0-h-1) = -1$$

$$\Rightarrow \quad f(0^+) \neq f(0^-),$$

therefore limit does not exist at $x=0$. Hence it is discontinuous at $x=0$

4.Sol: Given that $f(1)$ is increasing for values of x.

That is $f'(x)>0$, for all x.

$$\Rightarrow \quad f'(x)=\frac{k-2}{(\sin x+\cos x)^2}>0$$

$$\Rightarrow \quad k-2>0$$

i.e., $\quad k>2$.

5.Sol: Given $y=x+\dfrac{4}{x^2}$

$$\Rightarrow \frac{dy}{dx}=1-\frac{8}{x^3}$$

Also given slope of tangent is parallel to x-axis.

i.e., $\quad \dfrac{dy}{dx}=0$

$$\left(\frac{dy}{dx}\right)_{(x,y)}=1-\frac{8}{x_1^3}=0$$

$$\Rightarrow \quad x_1^3=8$$

i.e., $\quad x_1=2$ and $y_1=3$

The desired equations is $y-3=0(x-2)$

$$\Rightarrow y=3$$

6.Sol: Given that $\displaystyle\int\frac{1}{\sin x-\cos x+\sqrt{2}}dx$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\int\frac{1}{\left(\frac{1}{\sqrt{2}}\sin x-\cos x\frac{1}{\sqrt{2}}\right)+1}dx$$

$$\frac{1}{\sqrt{2}}\int\frac{1}{1-\cos\left(x+\frac{\pi}{4}\right)}dx$$

i.e., $$\frac{1}{\sqrt{2}}\int\frac{1}{2\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)}dx$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\int \cos ec^2x\left(\frac{x}{2}+\frac{\pi}{8}\right)dx$$

$$=\frac{1}{2\sqrt{2}}\frac{-\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)}{\frac{1}{2}}+c$$

$$=\frac{-1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$$

7.Sol: $I_{10}+I_8=\displaystyle\int_0^{\pi/4}\tan^8 x\left(\tan^2 x+1\right)dx$

$$=\left[\frac{\tan^9 x}{9}\right]_0^{\pi/4}=\frac{1}{9}$$

8.Sol: As coordinates of the triangle are rational. Therefore centroid of the triangle must be rational. And we know that centroid divides the line joining orthocenter and circumcenter in the ratio of 2 : 1.. The two points must also be rational.

9.Sol: Mid-points of diagonals are same.

Also, $S_{AB}=\dfrac{-4+1}{-2+4}=\dfrac{-3}{2}$, $S_{AD}=\dfrac{3+1}{2+4}=\dfrac{2}{3}$

$S_{AB}.S_{AD}=-1$. Hence it is a rectangle.

10.Sol: The given lines are

$$3x+4y=9 \quad (1)$$

and $$y=mx+1 \quad (2)$$

Solving (1) and (2), we get the x-coordinate of the point of intersection as $x=\dfrac{5}{4m+3}$.

Given that coordinates are inte

$\therefore \quad 4m+3=\pm 5$ or $4m+3=\pm 1$ intgers. That is x-coordinate is an integer.

Solving these, only integer values of m are -1 and -2

$\therefore \quad m=-1,-2$

11.Sol: Given circles are $x^2+y^2-y=0$ and $1\ x^2+y^2+y=0$, centres and radii of these circles are $C_1\left(0,\dfrac{1}{2}\right), C_2\left(0,-\dfrac{1}{2}\right)$ and $r_1=\dfrac{1}{2}, r_2=\dfrac{1}{2}$.

Now, $C_1C_2=\sqrt{0+\left(\dfrac{1}{2}+\dfrac{1}{2}\right)^2}=1$ and

$$r_1 + r_2 = \frac{1}{2} + \frac{1}{2} = 1$$

$\because \quad C_1C_2 = r_1 + r_2$

It means that two circles touch each externally. Hence, number of common tangents are 3.

12.Sol: Let $\left(2t, 2t^2\right)$ is the point on $24 = x^2$ that has shortest point from $(0,3)$. The slope of the tangent line at $\left(2t, 2t^2\right)$ is $2t$, this tangent must be perpendicular to the line connecting (0, 3) and $\left(2t, 2t^2\right)$.

so the product of slope is

$$2t\frac{\left(2t^2 - 3\right)}{2t} = -1$$

solving this equation, we get $t = \pm 1$ and the points are $(\pm 2, 2)$

13.Sol: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

It is given that it passes through $(7,0)$ and $(0,-5)$

i.e., $\quad a^2 = 49$ and $b^2 = 25$

Since $\quad b^2 = a^2\left(1 - e^2\right), \ \therefore 25 = 49\left(1 - e^2\right)$

$$\Rightarrow \ 1 - e^2 = \frac{25}{49} \Rightarrow e^2 = 1 - \frac{25}{49} = \frac{24}{49} \Rightarrow e = \frac{2\sqrt{6}}{7}$$

14.Sol: $P(a) = P(b) = P(c) = P(d) = 3$

$\Rightarrow \ P(x) = 3$ has a, b, c, d as its roots

$\Rightarrow P(x) - 3 = (x-a)(x-b)(x-c)(x-d)Q(x)$

[$\because Q(x)$ has integral coefficient]

Given $P(e) = 5$, then

$$(e-a)(e-b)(e-c)(e-d)Q(e) = 5$$

This is possible only when at least three of the five integers $(e-a), (e-b), (e-c), (e-d), Q(e)$ are equal to 1 or -1. Hence, two of them will be equal, which is not possible. Since a, b, c, d are distinct integers, $P(e) = 5$ is not possible.

15.Sol: Let

$$a^2(a+p) = b^2(b+p) = c^2(c+p) = \lambda \text{ (say)}$$

Then a, b, c are roots of $x^3 + px^2 - \lambda = 0$

$\therefore ab + bc + ca = 0$

16.Sol: Consider the identity:

$$(1+x)^n = P_0 + P_1x + P_2x^2 + P_3x^3 + ... + P_nx^n$$

Substitute $x = i$ on both sides.

$$(1+i)^n = P_0 + P_1i + P_2i^2 + P_3i^3 + ... + P_ni^n$$

$$\Rightarrow \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = (P_0 - P_2 + P_4 + ...)$$

$$+i(P_1 - P_3 + P_5 + ...)$$

$$\Rightarrow 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = (P_0 - P_2 + P_4 + ...)$$

$$+i(P_1 - P_3 + P_5 + ...)$$

Equate the real and imaginary parts, to get:

$$P_0 - P_2 + P_4 - P_6 + ... = 2^{n/2}\cos\frac{n\pi}{4} \text{ and}$$

$$P_1 - P_3 + P_5 - .... = 2^{n/2}\sin\frac{n\pi}{4}$$

17.Sol: Given expansion is

$${}^3C_0 - 5C_1 + 7C_2 - .... + (-1)^n(2n+3)C_n$$

$$= -2C_1 + 4C_2 - 6C_3 + ... + 2n(-1)^n C_n$$

$$+3\left(C_0 - C_1 + C_2 ... + (-1)^n C_n\right)$$

$$= -2\left(C_1 - 2C_2 + 3C_3 - .. + n(-1)^n C_n\right) + 3(0)$$

$$= -2(0) + 3(0) = 0$$

18.Sol: Since a person can alight at any one of n floors. Therefore, the number of ways in which m passengers can alight at n floors is

$$\underbrace{n \times n \times n \times ... \times n}_{m-\text{times}} = n^m.$$

The number of ways in which all passengers can alight at different floors is ${}^nC_m \times m! = {}^nP_m$

Hence, required probability $=\dfrac{{}^{n}P_{m}}{n^{m}}$

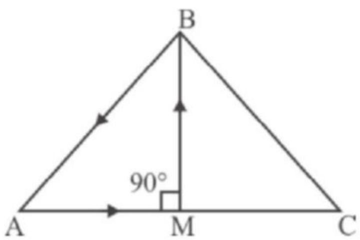
19.Sol: Given that a copies of one book, b copies of two books, c copies of three books, and 1 copy of d books, which yields total number of books in a shelf is

$a\times 1+b\times 2+3\times c+1\times d=a+2b+3c+d$.

The total number of ways these books can be arranged in a shelf is

$$\frac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$$

20.Sol: Since MB is the component of AB perpendicular to AC.



$AB=AB-AM$

$$=AB-\frac{(AB\cdot AC)AC}{(AC)^2}$$

$$=(6j-2k)-\frac{\{(6j-2k)\cdot(-5i+3j-k)\}(-5i+3j-k)}{(25+9+1)}$$

$$=(6j-2k)-\frac{20}{35}(-5i+3j-k)=\frac{10}{7}(2i+3j-k)$$

$$\therefore\quad BM=-\frac{10}{7}(2i+3j-k)$$

21.Sol: We have $\pi_{11}=-8-3+6=-8$

$$\pi_{22}=12-3-4-3=2$$

$$\therefore\qquad \pi_{11}\cdot\pi_{22}<0$$

The points lie on the opposite side of $\pi=0$.

22.Sol: Equation of the sphere concentric to given sphere will be $x^2+y^2+z^2-4x-2y-6z+d=0$

Since it passes through (0, 0, 0) then $d=0$

$\therefore$ Required equation of the sphere will be

$$x^2+y^2+z^2-4x-2y-6z=0.$$

23.Sol: Clearly, the roots of $x^2+\sqrt{2}x+1=0$ are non-real complex. So, one root common implies both roots are common.

So, $$\frac{a}{1}=\frac{b}{\sqrt{2}}=\frac{c}{1}=k.$$

$$\therefore\quad \cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{k^2+2k^2-k^2}{2\cdot k\cdot\sqrt{2}k}=\frac{1}{\sqrt{2}}.$$

24.Sol: Given $\tan A=4/3$, A is in the III quadrant

Now, $5\sin 2A+3\sin A+4\cos A$

$$=5(2\sin A\cos A)+3\sin A+4\cos A$$

$$=10\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)+3\left(-\frac{4}{5}\right)+4\left(-\frac{3}{5}\right)$$

$$=\frac{2A}{5}-\frac{12}{5}-\frac{12}{5}=0.$$

25.Sol: There are in total 4 elements and each elements can be chosen in exactly 3 ways. Hence, the number of all possible matrices of the said type is $3^4=81$.

26.Sol: Given $f(x)=\begin{vmatrix}x-3 & 2x^2-18 & 3x^3-81\\ x-5 & 2x^2-50 & 4x^3-500\\ 1 & 2 & 3\end{vmatrix}$

$$=(x-3)(x-5)\begin{vmatrix}1 & 2(x+3) & 3(x^2+3x+9)\\ 1 & 2(x+5) & 4(x^2+5x+25)\\ 1 & 2 & 1\end{vmatrix}$$

$$\Rightarrow\quad f(3)=f(5)=0$$

Now $$f(1)=\begin{vmatrix}-2 & -16 & -78\\ -4 & -48 & -496\\ 1 & 2 & 3\end{vmatrix}=2928$$

Therefore

$$f(1)\cdot f(3)+f(3)\cdot f(5)+f(5)\cdot f(1)=0=f(3)$$

27.Sol: $(p\vee q)\wedge(p\vee r)=p\vee(q\wedge r)$

28.Sol: Given that, mean of a set of numbers is $\overline{x}$. also given that each number is decreased by λ.

That is, required mean is $\dfrac{n\bar{x}-n\lambda}{n}=\bar{x}-\lambda$

29.Sol: $T_r=(-1)^r\dfrac{r^2+r+1}{r!}$

$$=(-1)^r\left[\frac{r}{(r-1)!}+\frac{1}{(r-1)!}+\frac{1}{r!}\right]$$

$$=(-1)^r\left[\frac{1}{(r-2)!}+\frac{1}{(r-1)!}+\frac{1}{(r-1)!}+\frac{1}{r!}\right]$$

$$=\left[\frac{(-1)^r}{r!}+\frac{(-1)^r}{(r-1)!}\right]+\left[\frac{(-1)^r}{(r-1)!}+\frac{(-1)^r}{(r-2)!}\right]$$

$$=\left[\frac{(-1)^r}{r!}-\frac{(-1)^{r-1}}{(r-1)!}\right]-\left[\frac{(-1)^{r-1}}{(r-1)!}-\frac{(-1)^{r-2}}{(r-2)!}\right]$$

$$=V(r)-V(r-1)$$

$$\therefore\ \sum_{r=1}^{n}T_r=V(n)-V(0)=\left[\frac{(-1)^n}{n!}-\frac{(-1)^{n-1}}{(n-1)!}\right]-1$$

Therefore, the sum of 20 terms is

$$\left[\frac{1}{20!}-\frac{-1}{19!}\right]-1=\frac{21}{20!}-1$$

30.Sol: Given $a\le f(x)\le b$ and f is strict decreasing (and hence invertible)

$$\Rightarrow f^{-1}(a)\ge f^{-1}(f(x))\ge f^{-1}(b)$$

$\Rightarrow f^{-1}(b)\le x\le f^{-1}(a)$ ($\because$ is decreasing if f^{-1} is decreasing)

Previous year JEE MAIN Questions

COORDINATE SYSTEMS & STRIAGHT LINES

[ONLINE QUESTIONS]

1. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle $30°$ with the positive direction of the x-axis, then the sum of the x-coordinates of the vertices of the square is : **[2017]**

(a) $2\sqrt{3}-1$ (b) $2\sqrt{3}-2$

(c) $\sqrt{3}-2$ (d) $\sqrt{3}-1$

2. A ray of light is incident along a line which meets another line, $7x-y+1=0$, at the point $(0,1)$. The ray is reflected from this point along the line, $y+2x=1$. Then the equation of the line of incidence of the ray of light is : **[2016]**

(a) $41x-25y+25=0$ (b) $41x+25y-25=0$

(c) $41x-38y+38=0$ (d) $41x+38y-38=0$

3. A straight line through origin O meets the lines $3y=10-4x$ and $8x+6y+5=0$ at points A and B respectively. Then O divides the segment AB in the ratio : **[2016]**

(a) 2 : 3 (b) 1 : 2 (c) 4 : 1 (d) 3 : 4

4. If a variable line drawn through the intersection of the lines $\frac{x}{3}+\frac{y}{4}=1$ and $\frac{x}{4}+\frac{y}{3}=1$, meets the coordinate axis at A and B, $(A \neq B)$, then the locus of the midpoint of AB is : **[2016]**

(a) $7xy=6(x+y)$

(b) $4(x+y)^2-28(x+y)+49=0$

(c) $6xy=7(x+y)$

(d) $14(x+y)^2-97(x+y)+168=0$

5. The point (2, 1) is translated parallel to the line $L: x-y=4$ by $2\sqrt{3}$ units. If the new points Q lies in the third quadrant, then the equation of the passing through Q and perpendicular to L is : **[2016]**

(a) $x+y=2-\sqrt{6}$ (b) $2x+2y=1-\sqrt{6}$

(c) $x+y=3-3\sqrt{6}$ (d) $x+y=3-2\sqrt{6}$

6. Let L be the line passing through the point $P(1, 2)$ such that its intercepted segment between the co-ordinate axis is bisected at P. If L_1 is the line perpendicualr to L and passing through the point $(-2,1)$, then the point of intersection of L and L_1 is : **[2015]**

(a) $\left(\frac{4}{5},\frac{12}{5}\right)$ (b) $\left(\frac{3}{5},\frac{23}{10}\right)$

(c) $\left(\frac{11}{20}, \frac{29}{10}\right)$ (d) $\left(\frac{3}{10}, \frac{17}{5}\right)$

7. The points $\left(0, \frac{8}{3}\right), (1,3)$ and $(82,30)$: **[2015]**

(a) From an acute angled triangle
(b) From a right angled triangle
(c) Lie on a striaght line.
(d) From an obtuse angled triangle

8. A straight line L through the point $(3,-2)$ is inclined at an angle of $60°$ to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, then the equation of L is : **[2015]**

(a) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$

(b) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

(c) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(d) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$

9. The circumcenter of a triangle lies at the origin and its centroid is the mid point of the line segment joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a), a \neq 0$. Then for any 'a' the orthocenter of this triangle lies on the line : **[2014]**

(a) $y - 2ax = 0$

(b) $y - (a^2 + 1)x = 0$

(c) $y + x = 0$

(d) $(a-1)^2 x - (a+1)^2 y = 0$

10. If a line intercepted between the coordinate axis is trisected at a point $A(4,3)$, which is nearer to x-axis, then its equation is : **[2014]**

(a) $4x - 3y = 7$ (b) $3x + 2y = 18$

(c) $3x + 8y = 36$ (d) $x + 3y = 13$

11. Given three points P, Q, R with $P(5, 3)$ and R lies on the x-axis. If equation of RQ is $x - 2y = 2$ and PQ is parallel to the x-axis, then the centroid of ΔPQR lies on the line : **[2014]**

(a) $2x + y - 9 = 0$ (b) $x - 2y + 1 = 0$

(c) $5x - 2y = 0$ (d) $2x - 5y = 0$

12. If a line L is perpendicular to the line $5x - y = 1$, and the area of the triangle formed by the line L and the coordinate axis is 5, then the distance of line L from the line $x + 5y = 0$ is : **[2014]**

(a) $\frac{7}{\sqrt{5}}$ (b) $\frac{5}{\sqrt{13}}$ (c) $\frac{7}{\sqrt{13}}$ (d) $\frac{5}{\sqrt{7}}$

13. If the three distinct lines $x + 2ay + a = 0, x + 3by + b = 0$ and $x + 4ay + a = 0$ are concurrent, then the point (a, b) lies on a : **[2014]**

(a) Circles (b) Hyperbola
(c) Straight line (d) Parabola

14. The base of an equilateral triangle is along the line given by $3x + 4y = 9$. If a vertex of the triangle is (1, 2), then the length of a side of the triangle is : **[2014]**

(a) $\frac{2\sqrt{3}}{15}$ (b) $\frac{4\sqrt{3}}{15}$ (c) $\frac{4\sqrt{3}}{5}$ (d) $\frac{2\sqrt{3}}{5}$

15. A light ray emerging from the point source placed at P (1, 3) is reflected at a point Q in the axis of x. If the reflected ray passes through the point R(6, 7), then the abscissa of Q is : **[2013]**

(a) 1 (b) 3 (c) $\frac{7}{2}$ (d) $\frac{5}{2}$

16. Let $A(-3,2)$ and $B(-2,1)$ be the vertices of a triangle ABC. If the centroid of this triangle lies on the line $3x + 4y + 2 = 0$, then the vertex C lies on the line : **[2013]**

(a) $4x + 3y + 5 = 0$ (b) $4x + 3y + 5 = 0$

(c) $3x + 4y + 3 = 0$ (d) $3x + 4y + 5 = 0$

17. If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the sides is $x = 2a$, then the area of the triangle, in square units, is : **[2013]**

(a) $\frac{5}{4}a^2$ (b) $\frac{5}{2}a^2$ (c) $\frac{25a^2}{4}$ (d) $5a^2$

18. If the x-intercept of some line L is double as that of the line, $3x + 4y = 12$ and the y-intercept of L is

half as that of the same line, then the slope of L is: **[2013]**

(a) -3 (b) $-3/8$ (c) $-3/2$ (d) $-3/16$

19. If the image of point P (2, 3) in a line L is Q (4, 5), then the image of point R (0, 0) in the same line is : **[2013]**

(a) (2, 2) (b) (4, 5) (c) (3, 4) (d) (7, 7)

20. Let θ_1 be the angle between two lines $2x+3y+c_1=0$ and $-x+5y+c_2=0$ and θ_2 be the angle between two lines $2x+3y+c_1=0$ and $-x+5y+c_3=0$, where c_1, c_2, c_3 are any real numbers : **[2013]**

Statement-1: If c_2 and c_3 are proportional, then $\theta_1=\theta_2$.

Statement-2: $\theta_1=\theta_2$ for all c_2 and c_3.

(a) Statement-1 is true, Statement-2 is true ; Statement-2 is a correct explanation of Statement-1.
(b) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation of Statement-1.
(c) Statement-1 is false ; Statement-2 is true.
(d) Statement-1 is true ; Statement-2 is false.

21. If the three lines $x-3y=p, ax+2y=q$ and $ax+y=r$ form a right-angled triangle then : **[2013]**

(a) $a^2-9a+18=0$ (b) $a^2-6a-12=0$

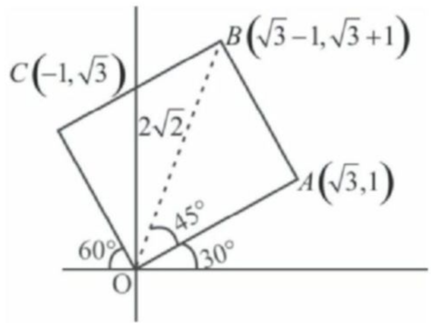
(c) $a^2-6a-18=0$ (d) $a^2-9a+12=0$

ANSWER KEY

1. b	2. c	3. c	4. a	5. d
6. a	7. c	8. c	9. d	10. b
11. d	12. b	13. c	14. b	15. d
16. c	17. b	18. d	19. d	20. a
21. a				

HINTS & SOLUTIONS

1.Sol: For A ; $\dfrac{x}{\cos 30^\circ}=\dfrac{y}{\sin 30^\circ}=2$



$\Rightarrow \quad x=\sqrt{3}$

and $\quad y=1$

For vertex C, $\dfrac{x}{\cos 120^\circ}=\dfrac{y}{\sin 120^\circ}=2$

$\Rightarrow \quad x=-1, y=\sqrt{3}$

For vertex B, $\dfrac{x}{\cos 75^\circ}=\dfrac{y}{\sin 75^\circ}=2\sqrt{2}$

$\Rightarrow \quad x=\sqrt{3}-1$

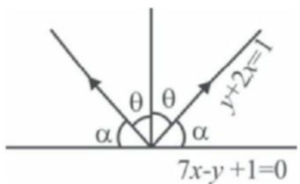
and $\quad y=\sqrt{3}+1$

$\therefore$ The sum of the x - coordinates is

$\sqrt{3}+\sqrt{3}-1-1=2\sqrt{3}-2$

2.Sol: Let slope of incident ray be m.

$\therefore$ angle of incidence = angle of reflection



$$\therefore \left|\frac{m-7}{1+7m}\right|=\left|\frac{-2-7}{1-14}\right|=\frac{9}{13}$$

$$\Rightarrow \frac{m-7}{1+7m}=\frac{9}{13} \text{ or } \frac{m-7}{1+7m}=-\frac{9}{13}$$

$\Rightarrow 13m-91=9+63$ or $13m-91=-9-63m$

$\Rightarrow 50m=-100$ or $76m=82$

$$\Rightarrow \quad m=-\frac{1}{2} \text{ or } m=\frac{41}{38}$$

$$\Rightarrow \quad y-1=-\frac{1}{2}(x-0) \text{ or } y-1=\frac{41}{38}(x-0)$$

i.e., $x+2y-2=0$ or $38y-38-41x=0$

$$\Rightarrow \quad 41x-38y+38=0$$

3.Sol: Length of perpendicular to $4x+3y=10$ from origin (0, 0) is

$$P_1=\frac{10}{5}=2$$

Length of perpendicular to $8x+6y+5=0$ from origin (0, 0) is

$$P_2=\frac{5}{10}=\frac{1}{2}$$

$\because$ Lines are parallel to each other $\Rightarrow$ ratio will be 4 : 1 or 1 : 4

4.Sol: Given that

$$L_1: 4x+3y-12=0$$

$$L_2: 3x+4y-12=0$$

i.e., $\quad L_1+\lambda L_2=0$

we have equation of a line passing through the intersection of given L_1

$$(4x+3y-12)+\lambda(3x+4y-12)=0$$

$$x(4+3\lambda)+y(3+4\lambda)-12(1+\lambda)=0$$

now, this line cuts x - axis at

$$A\left(\frac{12(1+\lambda)}{4+3\lambda},0\right)$$

now then line also cuts y - axis at

$$B\left(0,\frac{12(1+\lambda)}{3+4\lambda}\right)$$

Mid point (h, k) is $\left(\frac{1+\lambda}{4+3\lambda},\frac{6(1+\lambda)}{3+4\lambda}\right)$

Eliminating λ from h, and k, we get

$$6(h+k)=7hk$$

i.e., $\quad 6(x+y)=7xy$ is the locus

5.Sol: Slope of $x-y=4$ is $\tan\theta=1$

$$\Rightarrow \sin\theta\frac{1}{\sqrt{2}},\cos\theta=\frac{1}{\sqrt{2}} \text{ or}$$

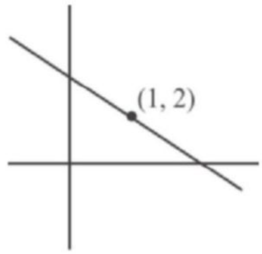
$$\sin\theta=\frac{-1}{\sqrt{2}},\cos\theta=\frac{-1}{\sqrt{2}}$$

i.e., $\quad Q=\left(2+2\sqrt{3}\sin\theta,1+2\sqrt{2}\cos\theta\right)$

$\therefore \quad =\left(2-\sqrt{6},1-\sqrt{6}\right)$

$\therefore$ The equation of required line is $x+y=3-2\sqrt{6}$

6.Sol: Equation of line L is



$$\frac{x}{2}+\frac{y}{4}=1$$

$$2x+y=4 \qquad (1)$$

For line

i.e., $\quad x-2y=-4 \qquad (2)$

Solving equation (1) and (2) ; we get point of intersection $\left(\frac{4}{5},\frac{12}{5}\right)$

7.Sol: Given that $A\left(0,\frac{8}{3}\right),B(1,2)C(89,30)$

Slope of $\quad AB=\frac{1}{3}$

Slope of $\quad BC=\frac{1}{3}$

So, the points A,B,C are collinear.

8.Sol: Given equation of line is $y+\sqrt{3}x-1=0$

$\Rightarrow \quad y=-\sqrt{3}x+1$

i.e., $\quad m=-\sqrt{3}$

Let the slope other line be m_1

we have $\tan 60^\circ = \left|\dfrac{m_1-(-\sqrt{3})}{1+(-\sqrt{3}m_1)}\right|$

$\Rightarrow \quad m_1 = 0, m_2 = \sqrt{3}$

given that line L is passing through $(3,-2)$

i.e., $\quad y-(-2) = +\sqrt{3}(x-3)$

$\Rightarrow \quad y+2 = \sqrt{3}(x-3)$

$y-\sqrt{3}x+2+3\sqrt{3}=0$

9.Sol: Given that and Circumcenter is $(0,0)$

Centroid $\left(\dfrac{(a+1)^2}{2}, \dfrac{(a-1)^2}{2}\right)$

We know the circumcenter (O),

Centroid (G) and orthocenter (H) are related as

$HG;GO = 2:1$

i.e., $\dfrac{HG}{GO} = \dfrac{2}{1}$

$\Rightarrow$ Coordinate of orthocenter is

$\left(\dfrac{3(a+1)^2}{2}, \dfrac{3(a-1)^2}{2}\right)$

Let $h = \dfrac{3}{2}(a+1)^2$ and $k = \dfrac{3}{2}(a-1)^2$

$\Rightarrow \dfrac{(a+1)^2}{h} = \dfrac{(a-1)^2}{k}$

$k(a+1)^2 = (a-1)^2 h$

$\therefore$ Locus is $y(a+1)^2 = (a-1)^2 x$

10.Sol: Given that A divider CB in 2:1

i.e., $4 = \left(\dfrac{1\times 0+2\times a}{1+2}\right)$ and $3 = \left(\dfrac{1\times b+2\times 0}{1+2}\right)$

$\Rightarrow \dfrac{2a}{3} = 4$ and $\dfrac{b}{3} = 3$

$\Rightarrow a = 6$ and $b = 9$

$\therefore$ The coordinates of B is (6, 0) and coordinates of A is (0, 9)

now, slope of line is $\dfrac{-3}{2}$

Equation of required line is $y-0 = \dfrac{-3}{2}(x-6)$

i.e., $3x+2y=18$

11.Sol: Equation of RQ is $x-2y=2$ (1)

Since, R is on x - axis

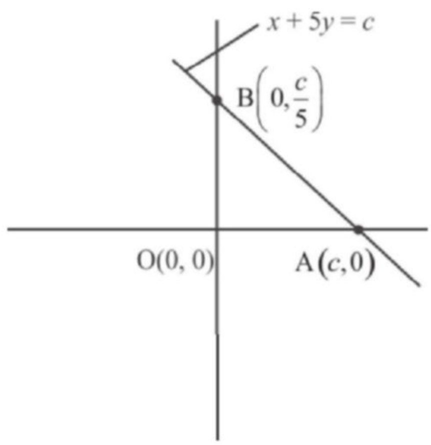
$x=2,\ y=0$

as PQ is parallel to x-*axis*, y-coordinates of Q is also 3 putting value of y in equation (1), we get Q $(8,3)$

Centroid of $\Delta PQR = \left(\dfrac{8+5+2}{3}, \dfrac{3+3}{3}\right) = (5,2)$

Only $(2x-5y=0)$ satisfy the given co-ordinates.

12.Sol: Let equation of line L, perpendicular to $5x-y=1$ be $x+5y=c$



Given that area of ΔAOB is 5.

i.e., $\quad 5 = \dfrac{1}{2}\left[c\left(\dfrac{c}{5}\right)\right]$

$\Rightarrow \quad c = \pm\sqrt{50}$

$\therefore$ Equation of line L is $x+5y=\pm\sqrt{50}$

Distance between L and line $x+5y=0$ is

$$d=\left|\frac{\pm\sqrt{50}-0}{\sqrt{1^2+5^2}}\right|=\frac{\sqrt{50}}{\sqrt{26}}=\frac{5}{\sqrt{13}}$$

13.Sol: Given that

$$x+2ay+a=0 \quad (1)$$

$$x+3by+b=0 \quad (2)$$

$$x+4ay+a=0 \quad (3)$$

are concurrent

i.e., $\begin{vmatrix}1 & 2a & a\\ 1 & 3a & b\\ 1 & 4a & a\end{vmatrix}=0$

$R_2 \to R_2-R_1$ and $R_3 \to R_3-R_1$

i.e., $\begin{vmatrix}1 & 2a & a\\ 0 & 3b-2a & b-a\\ 0 & 2a & 0\end{vmatrix}=0$

$\Rightarrow 2a=0$ or $b=a$

$\therefore$ Locus of $(a,b) \Rightarrow k=0$ or $y=x$

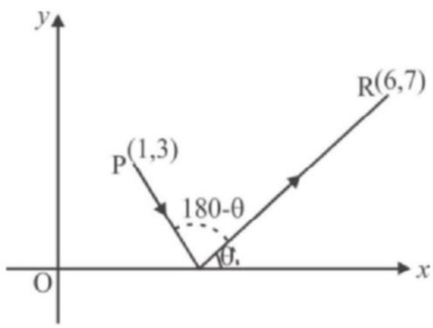
14.Sol: Since the point $(1, 2)$ does not lies on the base line $(3x+4y=9)$ i.e., the perpendicular distance from the paid to the base is altitude

$$\therefore h=\frac{|3(1)+4(2)-9|}{\sqrt{3^2+4^2}}=\frac{2}{5}$$

And we have $h=\frac{\sqrt{3}}{2}a$, where a is the side of an equilateral triangle

$$\therefore \quad a=\frac{2h}{\sqrt{3}}=\frac{\frac{4}{5}}{\sqrt{3}}=\frac{4\sqrt{3}}{15}$$

15.Sol:



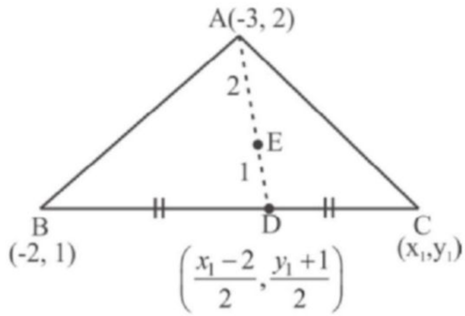
Let θ be $(x,0)$, then we have

$$\tan\theta=\frac{0-7}{x-6} \text{ and } \tan(180-\theta)=\frac{0-3}{x-1}$$

we have $\tan(180-\theta)=-\tan\theta$

i.e., $\frac{-3}{x-1}=\frac{+7}{x-6} \Rightarrow x=\frac{5}{2}$

16.Sol: Let $C=(x_1,y_1)$



Centroid, $E=\left(\frac{x_1-5}{3},\frac{y_1+3}{3}\right)$

Given that centroid lies on the line

$$3x+4y+2=0$$

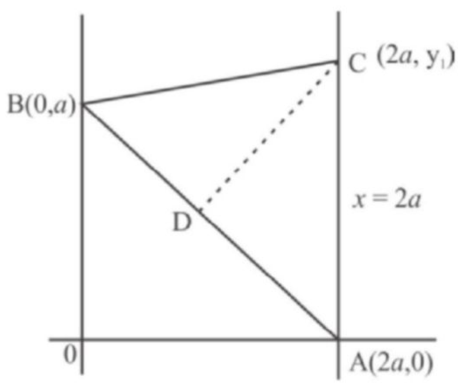
i.e., $3\left(\frac{x_1-5}{3}\right)+4\left(\frac{y_1+3}{3}\right)+2=0$

$$\Rightarrow \quad 3x_1+4y_1+3=0$$

Hence vertex (x_1,y_1) lies on the line

$$3x+4y+3=0$$

17.Sol:



Now $AB=\sqrt{(2a-0)^2+(0-a)^2}=\sqrt{5a^2}=15a$

We know D is midpoint of AB.

i.e., coordinates of D is $\left(\frac{2a+0}{2},\frac{0+a}{2}\right)=\left(a,\frac{a}{2}\right)$

we have BC = AC

i.e., $\sqrt{(2a-0)^2+(y_1-a)^2}=\sqrt{(2a-2a)^2+(y,-0)^2}$

$\Rightarrow 4a^2+(y_1-a)^2=(y_1-0)^2$

$\Rightarrow 5a^2-2ay_1=0$

$\Rightarrow y_1=\frac{5a}{2}$

Now $CD=\sqrt{(a-2a)^2+\left(\frac{5a}{2}-\frac{a}{2}\right)^2}$

$=\sqrt{a^2+4a^2}=15\,a.$

$\therefore$ Area of a triangle is $\frac{1}{2}AB\times CD$

i.e., $\frac{1}{2}\sqrt{5}a\times\sqrt{5}a=\frac{5a^2}{2}$

18.Sol: Given line $3x+4y=12$ can be written as

$$\frac{3x}{12}+\frac{4y}{12}=1\Rightarrow\frac{x}{4}+\frac{y}{3}=1$$

$\Rightarrow$ x-intercept $=4$ and y-intercept $=3$

Let the required line be

$L:\frac{x}{a}+\frac{y}{b}=1$ where

$a=$ x-intercept and $b=$ y-intercept

According to the question

$a=4\times2=8$ and $b=3/2$

$\therefore$ Required line is $\frac{x}{8}+\frac{2y}{3}=1$

$\Rightarrow \quad 3x+16y=24$

$\Rightarrow \quad y=\frac{-3}{16}x+\frac{24}{16}$

Hence, required slope $=\frac{-3}{16}$.

19.Sol: Lies on L

i.e., equation of a line is

$$(y-4)=\left(\frac{\frac{-1}{5-3}}{4-2}\right)(x-3)$$

$\Rightarrow y-4=-x+3$

$\therefore$ image of R(0, 0) with respect to the line $x+y-7=0$ is $(7,7)$

20.Sol: Two lines $-x+5y+c_2=0$ and $-x+5y+c_3=0$ are parallel to each other. Hence statement-1 is true, statement-2 is true and statement-2 is the correct explanation of statement-1.

21.Sol: Since three lines $x-3y=p$, $ax+2y=q$ and

$ax+y=r$

form a right angled triangle

i.e., product of slopes of any two lines $=-1$

Suppose $ax+2y=q$ and $x-3y=p$ perpendicular to each other.

$\therefore \quad \frac{-a}{2}\times\frac{1}{3}=-1\Rightarrow a=6$

Now, consider option one by one

$a=6$ satisfies only option (a)

$\therefore$ Required answer is $a^2-9a+18=0$

INTERNATIONAL CURRICULUM

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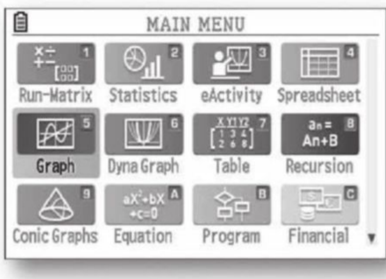
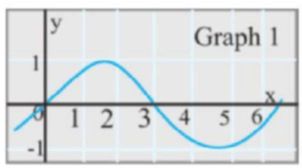
Understanding F'(x) curve

Technology partner: Casio , fx - Graphical Display calculator



Let $y = f(x)$ be a function

Let us consider $f(x) = \sin x$



I have taken a simple and standard function. The above graph represents $f(x) = \sin x$ drawn between $x = 0$ and $x = 2\pi$

we know $f'(x)$ means derivative of the function or it is a rate change function otherwise called as $\frac{dy}{dx}$ - instantaneous rate change of y with respect

to x for the curve $y = f(x)$ .The curve

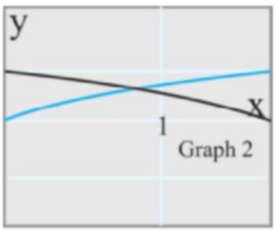
- $f(x)$ is increasing when $f'(x) > 0$
- $f(x)$ is decreasing when $f'(x) < 0$

Now let us connect the sign of $f'(x)$ to a graph when the derivative of $f(x)$ is positive (> 0) then the graph is ascending or increasing.What does it mean

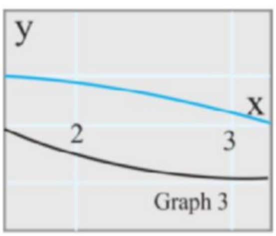
for $x_1 < x_2$ for $f(x_1) < f(x_2)$ or we call here as $\boxed{f'(x) > 0}$

When the derivative of (x) is negative (< 0) then the graph is descending or decreasing. What does it mean?

for $x_1 < x_2$ for $f(x_1) > f(x_2)$ or we call here as $\boxed{f'(x) < 0}$



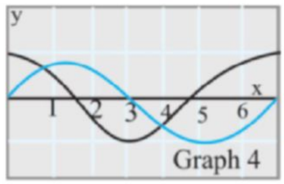
Now let us look at the sketch of $f(x)$ for the sub interval $0 \le x \le \frac{\pi}{2}$ in blue color. If you notice, $f(x)$ is increasing or ascending curve.The rate change or derivative is positive.A graph is positive when it is above x-axis .The black graph represents $f'(x)$ .For $0 \le x \le \frac{\pi}{2}$, $f'(x)$ graph is above x - axis



Now let us look at the sketch of $f(x)$ for the second sub interval $\frac{\pi}{2} \le x \le \pi$ in blue colour. if you notice,f (x) is decreasing or descending curve.The rate change or derivative is negative. A graph is negative when it is below x-axis. The black graph represent $f'(x)$. For

$\frac{\pi}{2} \le x \le \pi$, $f'(x)$ graph is below x-axis

Now I shall draw the full curve for $0 \le x \le 2\pi$ of both f(x) and $f'(x)$ on same graph. You can notice the above 2 concepts illustrated below



The blue graph is **increasing** for $0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} \le x \le 2$ in this interval if you notice black graph i,e $f'(x)$ graph is above x-axis

The blue graph is **decreasing** for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ in this interval if you notice black graph ie $f'(x)$ graph is below x-axis

Interpreting stationary point using $f'(x)$ graph

we know stationary points exist for a function *f(x)* when $f'(x) = 0$. looking at the *f(x)* in blue colour we know $f(x) = \sin x$ has a stationary maxima at $x = \frac{\pi}{2}$ and stationary minima at $x = \frac{3\pi}{2}$. in graph 4, we also have $f'(x)$ in black colour.Taking a closer look at it $f'(x) = 0$ at $x = \frac{\pi}{2}$ and at $x = \frac{3\pi}{2}$. At these 2points, the graph $f'(x)$ cut

the x-axis . So these are zeroes of the graph $f'(x)$ which is stationary of *f(x)*

How do we determine nature of stationary from $f'(x)$ graph?

At $x=\frac{\pi}{2}$ $f'(x)$ graph moves from positive to negative.By "first derivative test",*f(x)* has a mixima if the sign of $f'(x)$ changes from negative to positive .This is evident from the fact that graph of $f'(x)$ moves from above x-axis to below x- axis

At $x=\frac{3\pi}{2}$, $f'(x)$ graph moves from negative to positive .By"first derivative test",*f(x)* has a minima if the sign of $f'(x)$ changes from negative to positive.This is evident from the fact that graph of $f'(x)$ moves from below x-axis to above x-axis

soon you will hear

- Interpreting $f(x)$ from $f'(x)$ graph
- understanding $f''(x)$ graph
- Interpreting $f''(x)$ graph from $f'(x)$ graph

Absolute value - Inequalities

(1) $|x|<k$ where k is a non-zero positive real constant

we can $|x|<k$ in three ways

a. inequality
b. number line
c. interval

Inequality representation : $-k<x<k$
Number line representation :

interval representation : $x\in(-k,k)$ .The curve bracket indicates open interval .This means x is between -k and k but not equal to -k or k

(2) $|x|\le k$ where k is a non-zero real constant

inequality representation : $-k\le x\le k$
number line representation :

interval representation: $x\in[-k,k]$ .The box bracket indicates closed interval. This means x is between $-k$ and k inclusive of $-k$ and k

(3) $|x|>k$ where k is a non -zero real constant we can $|x|>k$ in three ways

a. inequality
b. number line
c. interval

inequality representative : $x<-kUx>k$
number line representation :

interval representation: $x\in[-\infty,-k]U(k,\infty)$. The curve bracket indicates open interval. This means x is between $-k$ and k but not equal to either of the limits

(4) $|x|\ge k$ where k is a non-zero real constant

Inequality representation: $x\le -k\ Ux\ge k$
Number line representation :

interval representation: $x\in(-\infty,-k]U[k,\infty)$ .The box bracket indicates closed interval .The open bracket indicates open interval .I mentioned that k has to be positive real constant and not merely real constant

Why k has to be positive real constant ?

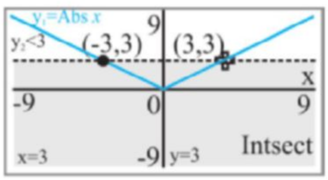
Let us revisit $|x|<k$

let us assume $k<0$ ie negative

if k is negative, how will $|x|$ will be less than a negative real constant , as the result of $|x|$ is always positive or zero(when $x=0$). As this contradicts ,our assumption $k<0$ fails. So k has to be a positive real constant

Graphs of inequalities

A sample problem: $|x|<3$



we sketch $|x|<3$ in casio graphical display calculator using the method of

First sketching $y=|x|$ and $y<3$

By inspecting the point of intersection we obtain the following

$|x|<3$ for $-3<x<3$

$|x|=3$ for $x=-3$ or $+3$

$|x|>3$ for $x<-3$ or $x>3$

Previous year JEE MAIN Questions

APPLICATIONS OF TRIGONOMETRY

[ONLINE QUESTIONS]

1. In a $\Delta ABC, \frac{a}{b} = 2+\sqrt{3}$ and $\angle C = 60^\circ$. Then the ordered pair $(\angle A, \angle B)$ is equal to :

[2015]

(a) $(45^\circ, 75^\circ)$ (b) $(105^\circ, 15^\circ)$

(c) $(15^\circ, 105^\circ)$ (d) $(75^\circ, 45^\circ)$

2. Let 10 vertical poles standing at equal distances on a straight line, subtend the same angle of elevation at a point *O* on this line and all the poles are on the same side of *O*. If the height of the longest pole is '*h*' and the distance of the foot of the smallest pole from *O* is '*a*' ; then the distance between two consecutive poles is :

[2015]

(a) $\frac{h\cos\alpha - \alpha\sin\alpha}{9\sin\alpha}$ (b) $\frac{h\sin\alpha - a\cos\alpha}{9\sin\alpha}$

(c) $\frac{h\cos\alpha - a\sin\alpha}{9\cos\alpha}$ (d) $\frac{h\sin\alpha - a\cos\alpha}{9\cos\alpha}$

3. The angle of elevation of the top of a vertical tower from a point P on the horizontal ground was observed to be α. After moving a distance 2 meters from P towards the foot of the tower, the angle of elevation changes to β. Then the height (in metres) of the tower is :

[2014]

(a) $\frac{2\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ (b) $\frac{\sin\alpha\sin\beta}{\cos(\beta-\alpha)}$

(c) $\frac{2\sin(\beta-\alpha)}{\sin\alpha\cos\beta}$ (d) $\frac{\cos(\beta-\alpha)}{\sin\alpha\sin\beta}$

[OFFLINE QUESTIONS]

1. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP=2AB. If $\angle BPC = \beta$, then $\tan\beta$ is equal to :

[2017]

(a) $\frac{4}{9}$ (b) $\frac{6}{7}$ (c) $\frac{1}{4}$ (d) $\frac{2}{9}$

2. A man is walking towards a vertical pillar in a straight at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minites from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, from B to reach the pillar, is :

[2016]

(a) 20 (b) 5 (c) 6 (d) 10

3. If the angles of elevation of the top of a tower from three collinear points A,B and C, on a line leading to the foot of the tower are $30^\circ, 45^\circ$ and 60° respectively. Then the ratio, AB : BC, is : **[2015]**

(a) $1:\sqrt{3}$ (b) $2:3$ (c) $\sqrt{3}:1$ (d) $\sqrt{3}:\sqrt{2}$

4. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed (in m/s) of the bird is **[2014]**

(a) $20\sqrt{2}$ (b) $20\left(\sqrt{3}-1\right)$

(c) $40\left(\sqrt{2}-1\right)$ (d) $40\left(\sqrt{3}-\sqrt{2}\right)$

5. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta, BC = p$ and $CD = q$, then AB is equal to : **[2013]**

(a) $\dfrac{\left(p^2+q^2\right)\sin\theta}{p\cos\theta + q\sin\theta}$ (b) $\dfrac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(c) $\dfrac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (d) $\dfrac{\left(p^2+q^2\right)\sin\theta}{\left(p\cos\theta+q\sin\theta\right)^2}$

ANSWER KEY

[ONLINE QUESTIONS]

1. b 2. a 3. a

[OFFLINE QUESTIONS]

1. d 2. b 3. c 4. b 5. a

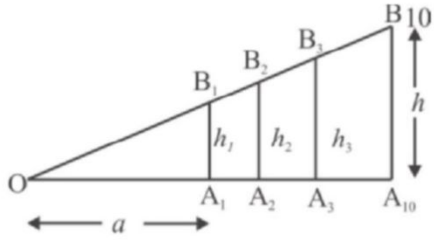
HINTS & SOLUTIONS

[ONLINE QUESTIONS]

1.Sol: $\dfrac{\sin A}{\sin B} = 2+\sqrt{3}$

$\dfrac{\sin\left(105^\circ\right)}{\sin\left(15^\circ\right)} = \dfrac{\cos 15^\circ}{\sin 15^\circ} = 2+\sqrt{3}$

2.Sol:



$\Delta OA_1B_1, \Delta OA_2B_2, \Delta A_3B_3, \ldots, \Delta OA_{10}B_{10}$ all are similar triangles.

$\Rightarrow \dfrac{h_1}{a_1} = \dfrac{h_2}{a_2} = \dfrac{h_3}{a_3} = \ldots = \dfrac{h_{10}}{a_{10}} = \tan\alpha.$

Since, $h_{10} = h = a_{10}\tan\alpha$ (1)

and $a_1 = a \Rightarrow h_1 = a\tan\alpha$ (2)

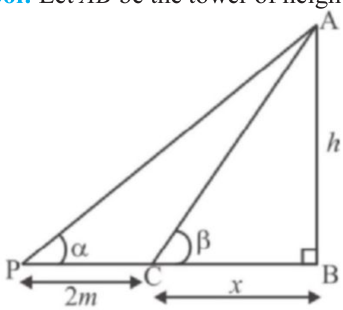
$\Rightarrow h = \left(a+9d\right)\tan\alpha$ where d is distance between poles $\left(\because a_{10} = a+9d\right)$

$\Rightarrow h = a\tan\alpha + 9d\tan\alpha$

$\Rightarrow \dfrac{h - a\tan\alpha}{9\tan\alpha} = d \Rightarrow \dfrac{h - \dfrac{a\sin\alpha}{\cos\alpha}}{9\dfrac{\sin\alpha}{\cos\alpha}} = d$

$\Rightarrow d = \dfrac{h\cos\alpha - a\sin\alpha}{9\sin\alpha}$

3.Sol: Let AB be the tower of height 'h'.



Given : In ΔABP $\tan\alpha = \dfrac{AB}{PB}$

or $\dfrac{\sin\alpha}{\cos\alpha} = \dfrac{h}{x+2}$

$$\Rightarrow (x+2)\sin\alpha = h\cos\alpha$$

$$\Rightarrow h = \frac{x\sin\alpha + 2\sin\alpha}{\cos\alpha}$$

Now, In $\Delta ABC, \tan\beta = \dfrac{AB}{BC}$

$$\Rightarrow \frac{\sin\beta}{\cos\beta} = \frac{h}{x} \Rightarrow x = \frac{h\cos\beta}{\sin\beta}$$

Putting the value of x in eq.(2) to eq. (1), we get

$$h = \frac{\dfrac{h\cos\beta\sin\alpha}{\sin\beta} + \dfrac{2\sin\alpha}{1}}{\cos\alpha}$$

$$\Rightarrow h = \frac{h\cos\beta\cdot\sin\alpha + 2\sin\alpha\sin\beta}{\sin\beta\cdot\cos\alpha}$$

$$\Rightarrow h(\sin\beta\cdot\cos\alpha - \cos\beta\cdot\sin\alpha) = 2\sin\alpha\cdot\sin\beta$$

$$\Rightarrow h[\sin(\beta-\alpha)] = 2\sin\alpha\cdot\beta$$

$$\Rightarrow h = \frac{2\sin\alpha\cdot\sin\beta}{\sin(\beta-\alpha)}$$

[OFFLINE QUESTIONS]

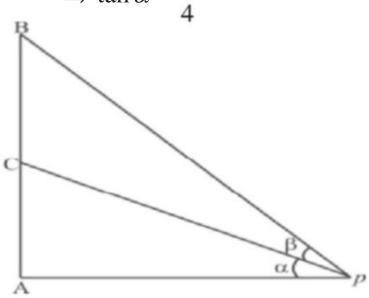
1.Sol: Since $AP = 2AB \Rightarrow \dfrac{AB}{AP} = \dfrac{1}{2}$

Let $\angle APC = \alpha$

$$\tan\alpha = \frac{AC}{AP} = \frac{1}{2}\frac{AB}{AP} = \frac{1}{4}$$

$$\left(\begin{array}{l} \because C \text{ is the midpoint} \\ \therefore AC = \dfrac{1}{2}AB \end{array}\right)$$

$$\Rightarrow \tan\alpha = \frac{1}{4}$$



As $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$\Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{2}\left[\begin{array}{l} \because \tan(\alpha+\beta) = \dfrac{AB}{AP} \\ \tan(\alpha+\beta) = \dfrac{1}{2}[\text{From (1)}] \end{array}\right]$$

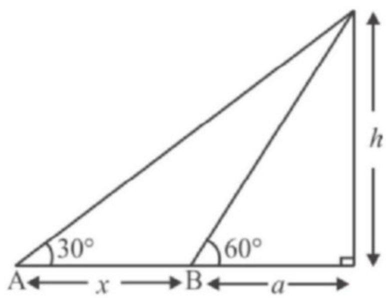
$$\Rightarrow \frac{\dfrac{1}{4} + \tan\beta}{1 - \dfrac{1}{4}\tan\beta} = \frac{1}{2}$$

$$\therefore \tan\beta = \frac{2}{9}$$

2.Sol: $\tan 30^\circ = \dfrac{h}{x+a} \Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{h}{x+a} \Rightarrow \sqrt{3}h = x+a$

(1)

$\tan 60^\circ = \dfrac{h}{a} \Rightarrow \sqrt{3} = \dfrac{h}{a} \Rightarrow h = \sqrt{3}a$



From (1) and (2)

$3a = x + a \Rightarrow x = 2a$

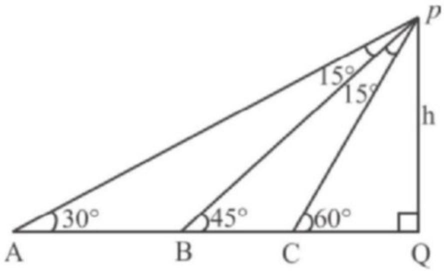
Here, the speed is uniform

So, time taken to cover $x = 2$ (time taken to cover a)

$\therefore$ Time taken to cover $a = \dfrac{10}{2}$ minutes =5 minutes

3.Sol: $\because$ PB bisects $\angle APC$, therefore

$AB : BC = PA : PC$



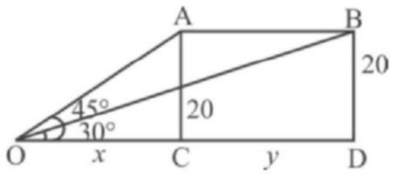
Also in $\Delta APQ, \sin 30^\circ = \frac{h}{PA} \Rightarrow PA = 2h$

and in $\Delta CPQ, \sin 60^\circ = \frac{h}{PC} \Rightarrow PC = \frac{2h}{\sqrt{3}}$

$\therefore \quad AB : BC = 2h : \frac{2h}{\sqrt{3}} = \sqrt{3} : 1$

4.Sol: Let the speed be m/sec.

Let AC be the vertical pole of height 20 m.



Let O be the point on the ground such that $\angle AOC = 45^\circ$

Let $\quad OC = x$

Time $\quad t = 1s$

From ΔAOC, $\tan 45^\circ = \frac{20}{x} \quad (1)$

and from ΔBOD, $\tan 30^\circ = \frac{20}{x+y} \quad (2)$

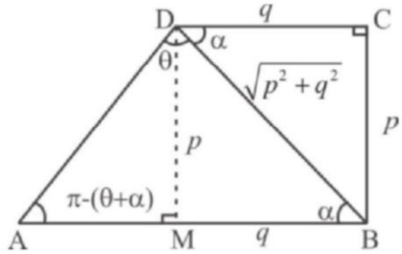
From (1) and (2), we have $x = 20$ and $\frac{1}{\sqrt{3}} = \frac{20}{x+y}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{20}{20+y} \Rightarrow 20 + y = 20\sqrt{3}$$

So, $\quad y = 20\left(\sqrt{3}-1\right)$

i.e., speed $= 20\left(\sqrt{3}-1\right) m/s$

5.Sol: From sine rule



$$\frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin\left(\pi-\left(\theta+\alpha\right)\right)}$$

$$AB = \frac{\sqrt{p^2+q^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha} = \frac{\left(p^2+q^2\right)\sin\theta}{q\sin\theta + p\cos\theta}$$

$$\left(\because \cos\alpha = \frac{q}{\sqrt{p^2+q^2}} \text{ and } \sin\alpha = \frac{p}{\sqrt{p^2+q^2}}\right)$$